Amenability of Certain Banach Algebras

CHENG Yin Hei

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Professor Ng Kung Fu (Chair)
Professor Leung Chi Wai (Thesis Supervisor)
Professor Luk Hing Sun (Committee Member)
Professor Huang Li Ren (External Examiner)

Abstract

The notion of amenability, is an important one in abstract harmonic analysis. B.E. Johnson showed that the amenability of a locally compact group can be characterized in terms of the Hochschild cohomology of its group algebra, thus initiated the theory of amenable Banach algebras. This survey is mainly focused on the problems relating the amenability of locally compact groups and the Banach algebras associated with the given groups. We will also see how the properties of these algebras and the relationships between them in the commutative case are generalized to the non-commutative case.

摘要

馴服性在抽象調和分析中是一個重要的概念。B.E. Johnson 證明了局部緊緻群的馴服性是隱藏於其群代數的 Hoschild 上同調中，這結果後來開創了馴服巴拿赫代數的理論。本文的重點在於局部緊緻群及其相關巴拿赫代數的馴服性問題，當中也會涉及到這些代數的性質和其相互關係如何從交換情形推廣至非交換的一般情形。

Introduction

A locally compact group G is said to be amenable if there is a left invariant mean on $L^\infty(G)$. If the axiom of choice is assumed, then we have the following strange phenomenon: "An orange can be cut into finitely many pieces, and these pieces can be reassembled to yield two oranges of the same size as the original one."
This is in fact an application of the so-called "paradoxical decompositions". One of the key ingredients for the proof of this paradox was that $\mathbb{F}_2$, the free group of two generators, lacks the property of amenability.

In the theory of abstract harmonic analysis, the group algebra $L^1(G)$ is undoubtedly the most important Banach algebra. It is thus very natural to ask how to use the group algebras to characterize the amenability of the given groups. In 1972, B.E. Johnson answered this question completely. He showed that G is amenable if and only if the first Hoschild cohomology group $\mathcal{H}(L^1(G), E^*) = \{0\}$ for any dual Banach $L^1(G)$-bimodule E. This condition makes sense for arbitrary Banach algebras, which may have nothing to do with locally compact groups. Consequently, it is meaningfully to define "amenable Banach algebras".

On the other hand, Eymard found that the Fourier algebra $A(G)$ is the "dual" object of the group algerba $L^1(G)$. If G is abelian, $A(G)$ is nothing but the image of the Fourier transform of $L^1(G)$. The name of the theory of $A(G)$ is therefore suggested as "non-commutative abstract harmonic analysis". As a Banach algebra, the most important property of the group algebra should be the existence of the bounded approximate identity. Many nice results on $L^1(G)$ in fact depend on this fact. The Fourier algebra, which is the dual object of the group algebra, unfortunately does not own a bounded approximate identity in general. In 1968, Leptin proved that $A(G)$ has a bounded approximate identity if and only if G is amenable. In the above point of view, amenable property becomes a reasonable assumption in the study of $A(G)$.

This survey article is divided into four chapters. All the basic results in the general theory of amenable Banach algebra will be presented in Chapter 1. The general properties of group algebras and measure algebras will be founded in Chapter 2. We will also see how the conditions on them characterize the amenability of a given group. As said above, the amenability assumption is a suitable one in the study of $A(G)$ when considering the "dual version" of the properties of $L^1(G)$. We will see that these "dual" properties are in fact equivalent to the amenable assumption on G. Another interesting question is to capture the amenability of a locally compact group G through the different sort of amenability of $A(G)$, which is called "operator amenability". In this case, the operator space structure of $A(G)$ is taken into account. Other operator cohomological properties are also considered in this chapter.

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Contents

Chapter 1

Preliminaries

We shall have some basic results in this chapter which will be used later.

1.1 Haar measures, Group algebras and measure algebras of locally compact groups

The sources of this chapter are [Fol][Chapter 2,3], [Run] [Appendix A] and [Dal] [Chapter 3] unless specified.

Definition 1.1.1. Let G be a locally compact group. A *Haar measure* on G is a non-zero Borel regular measure μ_G on G such that μ_G is *translation invariant*:

$$\text{(ie) } \mu_G(xE) = \mu_G(E) \text{ for any Borel set in } G$$

Theorem 1.1.2. *Let G be a locally compact group. Then there exists a Haar measure μ_G on G. Moreover, Haar measure on a locally compact group G exists uniquely up to a positive multiple.*

Proposition 1.1.3. *For a locally compact group G with a Haar measure μ_G, let U be an open subset in G, and let K be a compact subset in G. Then $\mu_G(U) > 0$ and $\mu_G(K) < \infty$.*

Theorem 1.1.4. *Let G be a locally compact group with a Haar measure μ_G. Then we have:*

(a) *G is compact if and only if $\mu_G(G) < \infty$*

(b) *G is discrete if and only if μ_G is a positive multiple of counting measure.*

Definition 1.1.5. Let G be a locally compact group with a Haar measure μ_G. If for any $x \in G$, we define $\mu_{xG}(E) = \mu_G(Ex)$, then $\mu_{xG}(E)$ is again a left Haar measure. By the uniqueness of Haar measures, there is a function $\Delta(x)$ such that $\mu_{xG} = \Delta(x)\mu_G$, and $\Delta(x)$ is independent of the oringinal choice of μ_G. The function $\Delta : G \longrightarrow (0, \infty)$ thus defined is called the *modular function* of G. Furthermore, G is said to be *unimodular* if $\Delta \equiv 1$.

Theorem 1.1.6. *Let G be a locally compact group. If G is compact or abelian, then G is unimodular.*

Theorem 1.1.7. *Let G be a locally compact group with a Haar measure μ_G. Then*

(a) *Δ is continuous homomorphism from G to $\mathbb{R}_\times$.*

(b) *For any $f \in L^1(\mu_G)$,*

$$\int_G f(xy)d\mu_G(x) = \Delta(y^{-1}) \int_G f(x)d\mu_G(x)$$

(c) *We have the following formula:*

$$\int_G f(x^{-1})\Delta(x^{-1})dm_G(x) = \int_G f(x)dm_G(x)$$

From now on, any group will be assumed to be locally compact, and G will denote a locally compact group. For any locally compact group, we fix a Haar measure m_G on G. For any compact group G, we choose the Haar measure m_G on G such that $m_G(G) = 1$. For any discrete group G, we choose m_G to be the counting measure.

Definition 1.1.8. Let $1 \leq p < \infty$ and let $\mathcal{L}^p(G)$ be the set of all p-integrable function on G with respect to m_G. Let $f_1, f_2 \in \mathcal{L}^p(G)$. f_1 and f_2 are said to be equivalent if $\|f_1 - f_2\|_p = 0$. Write $L^p(G)$ the set of all equivalent classes in $\mathcal{L}^p(G)$.

From now on, we use the following notation without further specification. Denote by

- $C_b(G)$ the space of bounded continuous functions on G.
- $C_0(G)$ the space of continuous functions vanishing at infinity on G.
- $C_c(G)$ the space of continuous functions with compact support on G.

Proposition 1.1.9. *If $1 < p < \infty$, then $C_C(G)$ is $\|\cdot\|_p$-dense in $L^p(G)$. Moreover, $C_c(G)$ is $\|\cdot\|_\infty$- dense in $C_0(G)$.*

Definition 1.1.10. Let $M(G)$ be the set of all complex Radon measure on G. Define a norm $\|\cdot\|_{M(G)}$ by

$$\|\mu\|_{M(G)} := |\mu|(G) \ \ (\mu \in M(G))$$

Also, define the convolution operation $*$ on $M(G)$ by

$$\int_G f(x)d(\mu * \nu)(x) := \int_G (\int_G f(xy)d\mu(x))d\nu(x) \ \ (f \in C_c(G), \mu, \nu \in M(G))$$

Definition 1.1.11. A measure in $M(G)$ is said to be *discrete* if it has the form $\mu = \sum_{s\in G} f(s)\delta_s$, where $f \in l^1(G)$. An element $\mu \in M(G)$ is called a *continuous measure* if $\mu(\{s\}) = 0$ $(s \in G)$.

Theorem 1.1.12. *The following statements are true for all locally compact group G:*

(a) *$(M(G), \|\cdot\|_{M(G)}, *)$ is a unital Banach *-algebra and the unit is given by the point mass measure at identity δ_e.*

(b) *The set $M_d(G)$ of discrete measures in $M(G)$ is a closed, unital *-subalgebra of $M(G)$ identified with $l^1(G)$; $M_d(G) = M(G)$ if and only if G is discrete.*

(c) *The set $M_c(G)$ of continuous measures in $M(G)$ is a closed *-ideal of $M(G)$, and $M(G) = M_c(G) \oplus_1 M_d(G)$.*

(d) *The set $M_a(G)$ of measures in $M(G)$ absolutely continuous with respect to the Haar measure is a closed *-ideal in $M(G)$ identified with $L^1(G)$ via*

$$\mu_f(E) := \int_E f(x)dm_G(x)$$

(e) *The convolution operation* $*$ *on* $L^1(G)$ *inherited from* $M(G)$ *is given by*

$$f * g(y) = \int_G f(yx)g(x^{-1})dm_G(x)$$

(f) *The set* $M_{cs}(G)$*, a subset of* $M_c(G)$*, consisting of the continuous measures which are singular with respect to the Haar measure is a closed *-subspace of* $M(G)$ *and* $M_c(G) = M_a(G) \oplus M_{cs}(G)$*;* $M_a(G) = M(G)$ *if and only if* G *is discrete.*

(g) $M(G) = M_a(G) \oplus M_{cs}(G) \oplus_1 M_d(G)$ *and suppose* $\mu \in M(G)$ *has the decomposition* $\mu = \mu_a + \mu_{cs} + \mu_d$*, then its total variation* $|\mu|$ *has the decomposition* $|\mu| = |\mu_a| + |\mu_{cs}| + |\mu_d|$.

Definition 1.1.13. Let X be a locally compact Hausdorff space X with a Radon measure μ. A set $E \subset X$ is *locally Borel* if $E \cap F$ is Borel whenever F is Borel and $\mu(F) < \infty$. A statement about points in X is true *locally almost everywhere* if it is true expect on a locally null set. A function $f : X \longrightarrow \mathbb{C}$ is *locally measurable* if $f^{-1}(A)$ is locally Borel for every Borel set $A \subset \mathbb{C}$. Define $L^\infty(\mu)$ to be the set of all locally measurable functions that are bounded except on a locally null set, modulo functions that are zero locally a.e. $L^\infty(\mu)$ is a Banach space with norm

$$\|f\|_\infty := \{c : |f| \leq c \text{ locally a.e.}\}$$

Remark. We have $(L^p(G))^* \cong L^q(G)$ if $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < \infty$, $(L^1(G))^* \cong L^\infty(G)$ and $(C_0(G))^* \cong M(G)$ from simple Banach space theory.

Theorem 1.1.14. *Suppose* $1 \leq p \leq \infty$, $f \in L^1(G)$, $g \in L^p(G)$. *Then the integral*

$$f * g(y) = \int_G f(yx)g(x^{-1})dm_G(x)$$

exists a.e. and we have $f * g \in L^p(G)$ *and* $\|f * g\|_p \leq \|f\|_1\|g\|_p$. *If* $p = \infty$*, then* $f * g \in C_b(G)$.

Definition 1.1.15. Let $1 \leq p \leq \infty$, $\mu \in M(G)$ and $f \in L^p(G)$. Define the function $\mu *_p f$ by

$$(\mu *_p f)(x) := \int_G f(y^{-1}x)d\mu(y)$$

If $f \in L^1(G)$, define the function $f * \mu$ by

$$(f * \mu)(x) := \int_G f(xy^{-1})\Delta(y^{-1})d\mu(y)$$

If $f \in L^\infty(G)$, define the function $f * \mu$ by

$$(f * \mu)(x) := \int_G f(xy^{-1})d\mu(y)$$

Remark. By definition 1.1.15, if $1 < p \leq \infty$, $\mu \in M(G)$ and $f \in L^p(G)$, then $L_{x^{-1}}f = \delta_x * f$; If $f \in L^1(G)$, then $\Delta(x)R_x f = f * \delta_x$ $(f \in L^1(G)$.

Theorem 1.1.16. *Suppose $\mu \in M(G)$, $g \in L^1(G)$. Then the function $\mu * f$ and $f * \mu$ exists a.e. and we have $\mu * f, f * \mu \in L^1(G)$ and $\|f * \mu\|_1$, $\|\mu * f\|_1 \leq \|\mu\|_{M(G)}\|f\|_1$. Thus, $(L^1(G), \|\cdot\|_1, *)$ is a closed ideal of $(M(G), \|\cdot\|_{M(G)}, *)$.*

Theorem 1.1.17. *If $1 \leq p < \infty$ and $f \in L^p(G)$, then*

$$\|L_y f - f\|_p,\ \|R_y f - f\|_p \to 0 \ as\ y \to e_G$$

Let f be a function on G. We say that f is *left*(resp. *right*) *uniformly continuous* if $\|L_y f - f\|_\infty \to 0$(resp. $\|R_y f - f\|_\infty \to 0$) as $y \to e_G$. If f is both left and right uniformly continuous, it is said to be *uniformly continuous.* Denote by $LUC(G)$ the space of left uniformly continuous functions on G, $RUC(G)$ the space of right uniformly continuous functions on G and $UCB(G)$ the space of uniformly continuous functions on G.

Theorem 1.1.18. *Let $\mathcal{B}$ be a the set of all open symmetric neighborhoods of e_G. Define a partial order on $\mathcal{B}$ by*

$$U_1 \preceq U_2 \ if\ U_2 \subseteq U_1$$

(a) *$\{\frac{1}{m_G(U)}\chi_U\}_{U \in \mathcal{B}}$ is an approximate identity of $L^1(G)$ which is self-adjoint with each term having L^1-norm equal to 1.*

(b) *If* $1 < p < \infty$, *then we have*

$$\frac{\chi_U}{m_G(U)} * f,\ f * \frac{\chi_U}{m_G(U)} \to f \ \in \ \|\cdot\|_p \ \ (f \in L^p(G))$$

(c) *If* $f \in RUC(G)$, *then*

$$f * \frac{\chi_U}{m_G(U)} \to f$$

(d) *If* $f \in LUC(G)$, *then*

$$\frac{\chi_U}{m_G(U)} * f \to f$$

(e) $\{\frac{1}{m_G(U)}\chi_U\}_{U\in\mathcal{B}}$ *is a bounded approximate identity of* $M(G)$ *in the weak sense:*

$$(ie) \quad \frac{\chi_U}{m_G(U)} * \mu,\ \mu * \frac{\chi_U}{m_G(U)} \to \mu \ in \ \sigma(M(G), C_0(G)) \ \ (\mu \in M(G))$$

Thus, $L^1(G)$ *is* $\sigma(M(G), C_0(G))$*-dense in* $M(G)$

Theorem 1.1.19. *Let G be a locally compact group. Then we have:*

(a) $LUC(G) = L^1(G) * L^\infty(G)$

(b) $RUC(G) = L^\infty(G) * L^1(G)$

(c) $UCB(G) = L^1(G) * L^\infty(G) * L^1(G)$

Theorem 1.1.20. *Let G, H be locally compact groups. Then*

$$L^1(G)\widehat{\otimes}L^1(H) \cong L^1(G \times H)$$

Theorem 1.1.21 (Wendel's theorem). *Let T be a left multiplier acting on $L^1(G)$. Then there exists a unique complex regular measure μ on G such that $T(F) = f * \mu$ for all f belonging to $L^1(G)$.*

Proof. Refer to [Wen] [Theorem 1]. □

Theorem 1.1.22. $\mathbb{T} \cdot \delta_G = \{T \in \mathfrak{M}(L^1(G), L^1(G)) : \|T(f)\|_1 = \|f\|_1 \ for\ any\ f \in L^1(G)\}$

Proof. Refer to [Wen] [Theorem 3]. □

Theorem 1.1.23. *Let $T : L^1(G_1) \longrightarrow L^1(G_2)$ be a norm-decreasing isomorphism. Then there exists a unique isomorphism $\tau : G_1 \longrightarrow G_2$ and a continuous character $\chi : G_1 \dashrightarrow \mathbb{C}$ such that*

(a) $TR_xT^{-1} = \chi(x^{-1})R_{\tau(x)}$ *for all x belonging to G*

(b) $(Tf)(\tau(x)) = c\tau(x)f(x)$ *for all x belonging to G*

where

$$\frac{m_{G_1}(E)}{m_{G_2}(\tau(E))}$$

for some Borel set E. Hence T is actually an isometry.

Proof. Refer to [Wen] [Theorem 5]. □

Corollary 1.1.24. *Let G_1, G_2 be locally compact groups. Then G_1 and G_2 are topologically isomorphic if and only if $L^1(G_1)$ and $L^1(G_2)$ are isometrically isomorphic.*

Theorem 1.1.25 (Johnson's theorem). *Let G_1, G_2 be locally compact groups. Then G_1 and G_2 are topologically isomorphic if and only if $M(G_1)$ and $M(G_2)$ are isometrically isomorphic.*

Proof. Refer to [Joh1] [Corollary of Theorem 3]. □

1.2 Banach algebras and amenability

All of the results in the following can be founded in [Run] [Chapter 2].

Definition 1.2.1. Let A be a Banach algebra and let E be a Banach space.

(a) A left(resp. right) A-module E is called a *left(resp. right) Banach A-module* if there is $K > 0$ such that

$$\|a \cdot x\| \leq k\|a\|\|x\| \quad (a \in A, x \in E)$$

$$(resp. \ \|a \cdot x\| \leq k\|a\|\|x\| \quad (a \in A, x \in E))$$

(b) An A-B-bimodule E is called a *Banach A-B-bimodule* if it is both a left Banach A-module and right Banach B-module.

(c) An A-bimodule E is called a *Banach A-bimodule* if it is both a left Banach A-module and right Banach A-module.

Definition 1.2.2. Let E, F be Banach A-bimodules. A map $\Phi : E \longrightarrow F$.

$$\Phi(u \cdot x) := u \cdot \Phi(x) \quad (u \in A, x \in E)$$

is said be a *bounded left multiplier* of E into F. Let $\mathfrak{M}_A(E, F)$ be the space of all bounded left multipliers of E into F. If A is a Banach algebra, A becomes a Banach A-bimodule in a natural manner. We simply write $\mathfrak{M}_A(A, A) := \mathfrak{M}(A, A)$.

Definition 1.2.3. Let A be a Banach algebra, and let E be a Banach A-bimodule.

(a) A bounded linear mapping $D : A \longrightarrow E$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A)$$

The set of all derivations from A into E is denoted by $\mathcal{Z}^1(A, E)$.

(b) Let $x \in E$. We define $ad_x : A \longrightarrow E, a \mapsto a \cdot x - x \cdot a$. It is easily seen that $ad_x \in \mathcal{Z}^1(A, E)$. Derivations of this form are called *inner derivations*. The set of all inner derivations is denoted by $\mathfrak{B}^1(A, E)$.

(c) The quotient space $\mathcal{H}^1(A, E) := \mathcal{Z}^1(A, E)/\mathfrak{B}^1(A, E)$ is called the *first Hochschild cohomology group* of A with coefficients in E.

Definition 1.2.4. Let A be a Banach algebra. If E is a Banach A-bimodule, then it is easily seen that E^* becomes a Banach A-bimodule through

$$< x, \phi \cdot a > := < a \cdot x, \phi >, \quad < x, a \cdot \phi > := < x \cdot a, \phi > \quad (a \in A, x \in E, \phi \in E^*)$$

The dual spaces of Banach bimodules equipped with these module operations are called *dual Banach bimodules*.

Proposition 1.2.5. *Let A be a Banach algebra, and let E, F be A-bimodules. Then $E \otimes F$ becomes an A-bimodule under the actions:*

$$a \cdot (x \otimes y) = (a \cdot x) \otimes y \text{ and } (x \otimes y) \cdot a = x \otimes (y \cdot a) \quad (a \in A, x \in E, y \in F)$$

Proposition 1.2.6. *Let A be a Banach algebra, and let I be its closed ideal. Then A/I becomes a Banach A-bimodule through*

$$b \cdot (a + I) = ba + I \text{ and } (a + I) \cdot b = ab + I \quad (a, b \in A)$$

Definition 1.2.7. Let A be a Banach algebra.

(a) If A is a unital Banach algebra, a Banach A-bimodule E is said to be *unital* if there exists $e_A \in A$ so that $e_A \cdot x = x \cdot e_A = x \quad (x \in E)$

(b) A Banach A-bimodule E is said to be *pseudo-unital* if $E = A \cdot E \cdot A$.

(c) A Banach left A-module E is said to be *essential* if $\overline{span(A \cdot E)}^{\|\cdot\|_E} = E$.

(d) A Banach A-bimodule E is said to be *essential* if $\overline{span(A \cdot E \cdot A)}^{\|\cdot\|_E} = E$.

Definition 1.2.8. Let E be a Banach A-bimodule. A *left(resp. right) approximate identity* in A for X is a net $\{e_\alpha\}_{\alpha \in A}$ in A such that

$$\|e_\alpha \cdot x - x\|_E \to 0 \;\; (x \in E)$$

$$(resp. \;\; \|e_\alpha \cdot x - x\|_E \to 0 \;\; (x \in E))$$

, and is said to be *bounded* if $sup_{\alpha \in A} \|e_\alpha\|_A < \infty$.
An *approximate identity* is both a left and a right approximate identity.

Theorem 1.2.9 (Cohen's factorization theorem). *If A has a bounded left(right) approximate identity for X, then $X = A \cdot X$ (resp. $X = X \cdot A$). In particular, if A has a bounded approximate identity for X, then $X = A \cdot X \cdot A$.*

Corollary 1.2.10. *Let A be a Banach algebra with a bounded approximate identity, and let X be a Banach A-bimodule. Then $A \cdot X$, $X \cdot A$ and $A \cdot X \cdot A$ are closed submodules of X.*

Proposition 1.2.11. *For a Banach algebra A with a bounded approximate identity, the followings are equivalent:*

(a) $\mathcal{H}^1(A, E^*) = \{0\}$ *for any Banach A-bimodule E.*

(b) $\mathcal{H}^1(A, E^*) = \{0\}$ *for any pseudo-unital Banach A-bimodule E.*

Definition 1.2.12. If a Banach algebra A is contained as a closed ideal in another Banach algebra B, then the *strict topology on B with respect to A* is defined through a family of semi-norms $(p_a)_{a \in A}$ where

$$p_a(b) := \|ba\| + \|ab\| \quad (b \in B).$$

The corresponding topology is denoted by $s(B, A)$.

Proposition 1.2.13. *Let A be a Banach algebra with a bounded approximate identity which is contained as a closed ideal in another Banach algebra B. Let E be a pseudo-unital Banach A-bimodule, and let $D \in Z^1(A, E^*)$. Then E is a Banach B-bimodule in a canonical fashion. Moreover, D has a unique extension $\tilde{D} \in \mathcal{Z}^1(B, E^*)$ which is continuous with respect to the strict topology $(s(B, A))$ on B and the w^*-topology on E^*.*

Definition 1.2.14. A Banach algebra A is called *amenable* if $\mathcal{H}^1(A, E^*) = \{0\}$ for any Banach A-bimodule E.

Proposition 1.2.15. *Let A be an amenable Banach algebra. Then A has a bounded approximate identity.*

Proposition 1.2.16. *Let A be a Banach algebra, and let E, F be A-bimodules. Then $E \otimes F$ becomes a Banach A-bimodule through*

$$a \cdot (x \otimes y) = (a \cdot x) \otimes y \text{ and } (x \otimes y) \cdot a = x \otimes (y \cdot a) \quad (a \in A, x \in E, y \in F)$$

From now on, $\widehat{\otimes}$ denotes the projective tensor product of Banach spaces.

Definition 1.2.17. If A is a Banach algebra, then the corresponding *diagonal operator* is defined through

$$\Delta_A : A\widehat{\otimes}A \longrightarrow A, a \otimes b \mapsto ab$$

Proposition 1.2.18. *Let A be a Banach algebra. $A\widehat{\otimes}A$ becomes a Banach A-bimodule through*

$$a \cdot (b \otimes c) := ab \otimes c \ \text{ and } \ (b \otimes c) \cdot a := b \otimes ca \ \ (a, b, c \in A)$$

Then Δ_A is a bimoule homomorphism with respect to this module structure.

Definition 1.2.19. Let A be a Banach algebra.

(a) An element $M \in (A\widehat{\otimes}A)^{**}$ is called a *virtual diagonal* for A if

$$a \cdot M = M \cdot a \ \text{ and } \ a \cdot \Delta_A^{**} M = a \ \ (a \in A)$$

(b) A bounded net $\{m_\alpha\}_{\alpha \in A} \subseteq (A\widehat{\otimes}A)$ is called an *approximate diagonal* for A if

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \ \text{ and } \ a\Delta_A m_\alpha \to 0 \ \ (a \in A)$$

Theorem 1.2.20. *For a Banach algebra A, the followings are equivalent:*

(a) *A is amenable.*

(b) *There is a virtual diagonal for A.*

(c) *There is an approxiamte diagonal for A.*

Proposition 1.2.21. *Let A be an amenable Banach algebra, let B be a Banach algebra, and let $\theta : A \longrightarrow B$ be a continuous homomorphism with dense range. Then B is amenable.*

Corollary 1.2.22. *If A is an amenable Banach algebra and if I is a closed ideal in A, then A/I is amenable.*

Definition 1.2.23. Let E be a Banach space. A closed subspace F of E is said to be *weakly complemented* in E if $F^{\perp}$ is complemented in E^{*}.

Theorem 1.2.24. *Let A be an amenable Banach algebra and let I be a closed ideal of A. Then the followings are equivalent:*

(a) *I is amenable.*

(b) *I has a bounded approximate identity.*

(c) *I is weakly complemented.*

Chapter 2

Cohomological properties of Group algebras and Measure algebras

2.1 Cohomological properties of $L^1(G)$

In this section, we shall see that the amenability of a group can be characterized by the vanishing of certain cohomology group of $L^1(G)$. Thus, we may use the cohomological triviality condition to define the amenability of general Banach algebras. We shall also characterize an amenable group G by some properties of the ideals of $L^1(G)$. Again, all of the proofs of the following results can be founded in [Run] [Chapter 2] unless specified.

Definition 2.1.1. Let G be a locally compact group. Let E be a subspace of $L^\infty(G)$. Suppose that E contains the constant functions and is closed under complex conjugation.

(a) E is said to be *left invariant* (reps. *right invariant*) if $\delta_g * \phi \in E$ (resp. $\phi * \delta_g \in E$) for all $\phi \in E$ and $g \in G$.

(b) A *mean* on E is a functional $m \in E^*$ such that $m(1) = \|m\| = 1$.

(c) If E is left invariant(resp. right invariant), then a mean m on E is said to be *left invariant* (resp. *right invariant*) if

$$< \delta_g * \phi, m >=< \phi, m > \quad (g \in G, \phi \in E)$$

$$(resp. < \phi * \delta_g, m >=< \phi, m > \quad (g \in G, \phi \in E))$$

(d) G is said to be amenable if there is a left invariant mean on $L^\infty(G)$.

Theorem 2.1.2. *Compact groups and locally compact abelian groups are amenable.*

Definition 2.1.3. Let G be a locally compact group.

(a) Let $L^1(G)^+_{\|\cdot\|=1} = \{f \in L^1(G) : f \leq 0 \text{ and } \|f\|_1 = 1\}$.

(b) Let E be a subspace of $L^\infty(G)$ such that $L^1(G)^+_{\|\cdot\|=1} * E \subseteq E$. A mean $m \in E^*$ is said to be *topologically left invariant* if

$$< f * \phi, m >=< \phi, m > \quad (\phi \in E, f \in L^1(G)^+_{\|\cdot\|=1})$$

Definition 2.1.4. Let G be a locally compact group. If π_1 and π_2 are two unitary representations of G. We say that π_1 is weakly contained in π_2 if $Ker(\pi_2^{C^*}) \subseteq Ker(\pi_1^{C^*})$ where π^{C^*} denotes the canonical representation on $C^*(G)$ corresponding to π.

Theorem 2.1.5. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *G is amenable.*

(b) *There is a left invariant mean on $C_b(G)$.*

(c) *There is a left invariant mean on $LUC(G)$.*

(d) *There is a left invariant mean on $RUC(G)$.*

(e) *There is a left invariant mean on $UC(G)$.*

(f) *There is a topologically left invariant mean on $UC(G)$.*

(g) *There is a topologically left invariant mean on $L^\infty(G)$.*

(h) *There is a net $(f_\alpha)_{\alpha\in A}$ in $L^1(G)^+_{\|\cdot\|=1}$ such that $\|\delta_g * f_\alpha - f_\alpha\|_1 \to 0 \quad (g \in G)$.*

(i) *There is a net* $\{f_\alpha\}_{\alpha\in A}$ *in* $L^1(G)^+_{\|\cdot\|=1}$ *such that* $\|\widetilde{f}*f_\alpha - f_\alpha\|_1 \to 0 \quad (\widetilde{f}\in L^1(G)^+_{\|\cdot\|=1})$

(j) *If* G *acts affinely on a compact, convex subset* K *of a locally compact convex space* E,

$$(i.e.)\quad g\cdot(tx+(1-t)y) = t(g\cdot x)+(1-t)(g\cdot y) \quad (g\in G, x,y\in K, t\in[0,1])$$

such that $G\times K \Longrightarrow K$, $(g,x)\mapsto g\cdot x$ *is separately continuous, then there is* $x\in K$ *such that* $g\cdot x = x\ \forall g\in G$.

(k) *Every irreducible unitary representation of* G *is weakly contained in* λ_2.

(l) *The trivial representation of* G *on* $\mathbb{C}$ *is weakly contained in* λ_2.

(m) *There exists* $p\in\mathbb{N}$ *such that for any compact subset* K *in* G *and* $\epsilon>0$, *there is* $f\in L^p(G)$ *with* $\|f\|_p = 1$ *such that*

$$\|\delta_x * f - f\| < \epsilon \quad (x\in K)$$

(P_p condition)

(n) *For any* $p\in\mathbb{N}$ *and for any compact subset* K *in* G *and* $\epsilon>0$, *there is* $f\in L^p(G)$ *with* $\|f\|_p=1$ *such that*

$$\|\delta_x * f - f\| < \epsilon \quad (x\in K)$$

(o) *If* $f\in L^1(G)^+$, *then* $\|\lambda_p(f)\| = \|\phi\|_1$ *for some* $p\in\mathbb{N}$.

(p) *If* $f\in L^1(G)^+$, *then* $\|\lambda_p(f)\| = \|\phi\|_1$ *for all* $p\in\mathbb{N}$.

(q) $C^*_r(G) = C^*(G)$

Theorem 2.1.6. *Let* G *be an amenable group. If* H *be a closed subgroup of* G, *then* H *is also amenable. If* N *is a closed normal subgroup of* G, *then* G/N *is amenable.*

Theorem 2.1.7 (Johnson's amenability theorem). *Let* G *be a locally compact group. Then* G *is amenable if and only if* $L^1(G)$ *is amenable.*

Proof. Refer to [Joh2] [Theorem 2.5]. □

Definition 2.1.8. A Banach algebra A is said to be *weakly amenable* if $\mathcal{H}^1(A, A^*) = \{0\}$

Theorem 2.1.9. *Let G be a locally compact group. Then $L^1(G)$ is weakly amenable.*

Proof. Refer to [D-G] [Theorem 1] or [Joh3]. □

Definition 2.1.10. A Banach algebra A is *biprojective* if $\Delta : A\widehat{\otimes}A \longrightarrow A$ has a bounded right inverse which is an A-bimodule homomorphism.

The relationship between biprojectivity and amenability is given by the following lemma.

Lemma 2.1.11. *Let A be a Banach algebra with bounded approximate identities. If A is biprojective, then A is amenable.*

Proof. Refer to [Dal] [Proposition 3.3.20]. □

Theorem 2.1.12. *Let G be a locally compact group. Then G is compact if and only if $L^1(G)$ is biprojective.*

Proof. Refer to [Dal] [Proposition 3.3.32] or [Run] [Example 4.3.2], [Lemma 4.3.10] and [Exercise 4.3.11]. □

2.2 Amenability and weak amenability of $M(G)$

It is well known that two groups are topological isomorphic if and only if their measure algebras are isometric isomorphic. As a result, there is a 1-1 correspondence between the categories of locally compact groups and measure algebras. It is thus reasonable that the amenability properties of a locally compact group and those of its own measure algebra are closely related. It is proven that the measure algebra of a locally compact group is amenable as a Banach algebra if and only if the group is discrete and amenable. It is also true that the measure algebra is weakly amenable if and only if the group is discrete. In this section, we shall sketch the proofs of these two theorems. They are based on [D-G-H].

Lemma 2.2.1. *Let G be a non-discrete, locally compact group. Then $\overline{M_c(G)}$ has infinite codimension in $M_c(G)$.*

Proof. The proof can be found in [D-G-H] [Theorem 2.7]. □

Theorem 2.2.2. *Let G be a locally compact group. Then G is discrete and amenable if and only if $M(G)$ is amenable.*

Proof. Assume that G is discrete, we have $M(G) \cong L^1(G)$ as Banach algebras. Therefore $M(G)$ is amenable by theorem 2.1.7. Conversely, if $M(G)$ is amenable, then $M_C(G)$ is amenable since $M_c(G)$ is a complemented ideal in $M(G) = M_c(G) \oplus M_s(G)$. Therefore, there exists a bounded approximate identity for $M_c(G)$. By Cohen's factorization theorem, we get $M_c(G)^2 = M_c(G)$. If G is not discrete, then $\overline{M_c(G)}$ has infinite codimension in $M_c(G)$ by lemma 2.2.1, which contradicts $(*)$. Therefore G is discrete, and hence $M(G) \cong L^1(G)$ is amenable, so G is amenable. □

Definition 2.2.3. A Banach algebra A is said to have *weak factorization* if $\overline{span(A^2)} = A$

Proposition 2.2.4. *A weakly amenable Banach algebra A always has weak factorization.*

Proof. See [B-C-D] □

Let A be a Banach algebra. Let ϕ be a character on A. It is easy to see that $\mathbb{C}$ is an A-bimodule under the products

$$a \cdot z = z \cdot a = \phi(a)z \;\; (a \in A, z \in C)$$

This one-dimensional bimodule is denoted by $\mathbb{C}_\phi$.

Definition 2.2.5. A derivation from A into $\mathbb{C}_\phi$ is called a *point derivation* at ϕ if it is a linear functional d on A such that

$$d(ab) = \phi(a)d(b) + \phi(b)d(a) \;\; (a, b \in A)$$

Proposition 2.2.6. *There is no non-zero, continuous point derivations on a weakly amenable Banach algebra.*

Proof. Let $d : A \longrightarrow \mathbb{C}_\phi$ be a point derivation of A at ϕ and define a derivation d_ϕ : $A \longrightarrow A^*$ by $d_\phi(a) = d(a)\phi$. Since A is weakly amenable, there exists $\psi \in A^*$ such that $d(a)\phi = \psi \cdot a - a \cdot \psi$. For any $x \in A$, we have

$$d(a)\phi(x) = \psi(ax) - \psi(xa)$$

Thus, for any $a \in A$, we have

$$d(a)\phi(a) = \psi(a^2) - \psi(a^2) = 0$$

By the definition of d, $d(a^2) = 2\phi(a)d(a) = 0$. Since A has weak factorization, it follows that $d = 0$. □

Definition 2.2.7. Let $\lambda \in M(G)^*$, $s \in G$. Define $s \cdot \lambda$ and $\lambda \cdot s$ by

$$\langle \mu, s \cdot \lambda \rangle = \langle \mu * \delta_s, \lambda \rangle \quad \textit{and} \quad \langle \mu, \lambda \cdot s \rangle = \langle \delta_s * \mu, \lambda \rangle \quad (\mu \in M(G))$$

Clearly,$s \cdot \lambda$ and $\lambda \cdot s \in M(G)$ and $\|s \cdot \lambda\| = \|\lambda \cdot s\| = \|\lambda\|$. The functional λ is said to be *translation invariant* if $s \cdot \lambda \cdot t = \lambda$ $(s, t \in G)$

Definition 2.2.8. By the decomposition $M(G) = M_c(G) \oplus M_s(G)$, let $\phi : M(G) \longrightarrow \mathbb{C}$, $\sum_{s\in G} \alpha_s \delta_s \mapsto \sum_{s \in G} \alpha_s$. Then ϕ is called the *discrete augmentation character* on $M(G)$.

Lemma 2.2.9. *Let G be a non-discrete, locally compact group. Then there exists a non-zero, translation invariant linear functional F in $M(G)^*$ such that*

(a) $F|_{M_d(G)} = 0$

(b) $F|_{M_c(G)^2} = 0$

Proof. The proof can be found in [D-G-H] [Theorem 3.1]. □

Theorem 2.2.10. *Let G be a locally compact group. Then the followings are equivalent:*

(a) *G is discrete.*

(b) *$M(G)$ is weakly amenable.*

(c) *$M(G)$ has no non-zero, continuous point derivation at ϕ.*

Proof. "(a) $\Rightarrow$ (b)" If G is discrete, then $M(G) \cong L^1(G)$. Therefore, $M(G)$ is weakly amenable by theorem 2.1.9.

"(b) $\Rightarrow$ (c)" See proposition 2.2.6.

"(c) $\Rightarrow$ (a)" Let F be as in lemma 2.2.9. It is sufficient to show that F is a point derivation at ϕ. Note that $F|_{M_d(G)} = F|_{M_c(G)^2} = 0$ and F is translation invariant. Take $\mu \in M_c(G)$, $\nu = \sum_{s\in G} \alpha_s \delta_s \in M_d(G)$, then

$$F(\mu * \nu) = \sum_{s\in G} \alpha_s F(\mu * \delta_s) = \sum_{s\in G} \alpha_s F(s \cdot \mu) = \sum_{s\in G} \alpha_s F(\mu) = \phi(\nu)F(\mu)$$

Similarly, $F(\nu * \mu) = F(\mu)\phi(\nu)$.

Therefore, for any $\mu_1, \mu_2 \in M_c(G), \nu_1, \nu_2 \in M_d(G)$,

$F((\mu_1 + \mu_2) * (\nu_1 + \nu_2)) = F(\mu_1 * \mu_2) + F(\mu_1 * \nu_2) + F(\mu_2 * \nu_1) + F(\nu_1 * \nu_2)$

$= \phi(\nu_1)F(\mu_2) + F(\mu_1)\phi(\nu_2) = \phi(\mu_1 + \nu_1)F(\mu_2 + \nu_2) + F(\mu_1 + \nu_1)\phi(\mu_2 + \nu_2)$

Therefore, F is a point derivation at ϕ. $\square$

Chapter 3

Amenability and Fourier algebras with Bounded Approximate identities

Throughout this chapter, let G be a locally compact group. We know that the Fourier algebra of G can be regarded as the "dual" object of its group algebra. However, in certain cases, the Fourier algebra does not really behave like the group algebra. The major difference between them is that the group algebra always has a bounded approximate identity while the Fourier algebra does not. The proof of many important results for the group algebra depends on the existence of bounded approximate identity. As we will see in this section, Leptin proved that $A(G)$ has a bounded approximate identity if and only if G is amenable. Thus, many important results for the group algebra can be transplanted easily to the Fourier algebra in the amenable case. Surprisingly, these results are in fact equivalent to the amenability of G.

3.1 Properties of Fourier-Stieltjes algebras, Fourier algebras, group von Neumann algebras

The following results are quoted for future convenience. Their proof are omitted and may be founded in [Eym] unless specified.

Definition 3.1.1. Define $P(G) := \{f \in C_b(G) : \int_G f(g^* * g) \geq 0 \ \forall g \in L^1(G)\}$. The elements in $P(G)$ are said to be positive definite function on G.

Definition 3.1.2. A *unitary representation* of G is a homomorphism π from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on some non-zero Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology.

Definition 3.1.3. Let G be a locally compact group.

(a) Let Σ_G be the set of all equivalence classes of unitary representations of G and $\widehat{G}$ be the subclass of all irreducible representations of G.

(b) Let $\lambda_2 : G \longrightarrow \mathfrak{L}(L^2(G))$, $\lambda_2(x)(g) := \delta_x * g$ $(g \in L^2(G))$ be the *left regular representation* of G.

(c) For any $f \in L^1(G)$, define

$$\|f\|_{C^*(G)} := sup_{\pi \in \Sigma_G} \|\pi(f)\|$$

It is easily seen that $\|\cdot\|_{C^*(G)}$ is a norm on $L^1(G)$. Let $C^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C^*(G)}$.

(d) Let $VN(G)$ be the von Neumann algebra generated by $\lambda_2(G)$ in $\mathfrak{L}(L^2(G))$. It is called the *group von Neumann algebra* of G.

(e) For any $f \in L^1(G)$, define

$$\|f\|_{C_r^*} := \|\lambda_2(f)\|$$

It is easily seen that $\|\cdot\|_{C^*_{\lambda_2}(G)}$ is a norm on $L^1(G)$. Let $C_r^*(G)$ be the completion of $L^1(G)$ under $\|\cdot\|_{C_r^*(G)}$.

Remark. By theorem 1.1.17, λ_2 is really a unitary representation of G.

From now on, write

- $B(G) := \{x \mapsto < \pi(x)\xi, \eta >: \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}$
- $A(G) := \{x \mapsto < \lambda_2(x)\xi, \eta >: \xi, \eta \in L^2(G)\}$.

For any function f on G, write

$$f^\vee(x) := f(x^{-1}) \text{ and } \tilde{f}(x) := \overline{f(x^{-1})} \quad (x \in G)$$

Theorem 3.1.4. *Let G be a locally compact group. For any $f \in B(G)$, define*

$$\|f\|_{B(G)} := inf\{\|\xi\|\|\eta\| : f(x) =< \pi(x)\xi, \eta > \quad where \ \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}$$

Then the following statements are true:

(a) $\|\cdot\|_{C^*(G)}$ *and* $\|\cdot\|_{C^*_{\lambda_2}(G)}$ *are C^*-norms and* $\|\cdot\|_{C^*(G)} \leq \|\cdot\|_{C^*_{\lambda_2}(G)} \leq \|\cdot\|_{L^1(G)}$ *on* $L^1(G)$

(b) $B(G) = span(P(G))$

(c) $(B(G), \|\cdot\|_{B(G)})$ *is a *-Banach algebra with pointwise multiplication and complex conjugation.*

(d) $\|f\|_{B(G)} = sup\{|\int_G fg| : g \in L^1(G), \|g\|_{C^*(G)} \leq 1\}$

(e) $(B(G), \|\cdot\|_{B(G)}) \cong (L^1(G), \|\cdot\|_{C^*(G)})^*$ *via*

$$< f, g >_{(L^1(G), B^1(G))} = < \pi_{L^1(G)}(f)\xi, \eta >= \int_G f(x)g(x)dm_G(x)$$

$$(f \in L^1(G), g \in B(G), g(x) \ = < \pi_G(x)\xi, \eta >, \xi, \eta \in \mathcal{H}_\pi)$$

(f) $(B(G), \|\cdot\|_{B(G)} \cong (C^*(G), \|\cdot\|_{C^*(G)})^*$

$$< f, g >_{(C^*(G), B^1(G))} = < \pi_{C^*(G)}(f)\xi, \eta >$$

$$(f \in C^*(G), g \in B(G), g(x) \ = < \pi_G(x)\xi, \eta >, \xi, \eta \in \mathcal{H}_\pi)$$

(g) *$B(G)$ is a translation invariant (generally non-closed)*-subalgebra of $L^\infty(G)$ and* $\|\cdot\|_{B(G)} \geq \|\cdot\|_{L^\infty(G)}$.

Theorem 3.1.5. *Let $u \in B(G)$, $x \in G$. Then*

$$\|u\|_{B(G)} = \|u^\vee\|_{B(G)} = \|\tilde{u}\|_{B(G)} = \|L_x u\|_{B(G)} = \|R_x u\|_{B(G)}$$

Theorem 3.1.6. *Let G be a locally compact group. For any $f \in A(G)$, define*

$$\|f\|_{A(G)} := inf\, \{\|\xi\|\|\eta\| : f(x) =< \lambda_2(x)\xi, \eta > \quad where\ \xi, \eta \in L^2(G)\}$$

Then the following statements hold:

(a) $\|\cdot\|_{B(G)} = \|\cdot\|_{A(G)}$ *on* $A(G)$.

(b) $A(G) = L^2(G) * \widetilde{L^2(G)}$

(c) $(A(G), \|\cdot\|_{A(G)})$ *is a closed ideal in* $(B(G), \|\cdot\|_{B(G)})$.

(d) $A(G)^* \cong VN(G)$ *via*

$$< \xi * \tilde{\eta}, T >_{(A(G),VN(G))} := < T\xi, \eta > \quad (\xi, \eta \in L^2(G), T \in VN(G))$$

and

$$< \xi * \tilde{\eta}, \lambda_2(x) >_{(A(G),VN(G))} = < \lambda_2(x)\xi, \eta > = \xi * \tilde{\eta}(x^{-1}) \quad (\xi, \eta \in L^2(G), x \in G)$$

(e) *$A(G)$ is a translation invariant *-subalgebra(generally non-closed) of $L^\infty(G)$ and* $\|\cdot\|_{A(G)} \geq \|\cdot\|_{L^\infty(G)}$.

(f) *$A(G)$ is the $\|\cdot\|_{B(G)}$-closure of $B(G) \cap C_c(G)$ in $B(G)$.*

(g) *$\sigma(A(G)) = G$ and hence $A(G)$ is semi-simple.*

Definition 3.1.7. (a) $B(G)$ is called the *Fourier Stieltjes algebra* of G.

(b) $A(G)$ is called the *Fourier algebra* of G.

(c) $C^*(G)$ is called the *full group C^*algebra* or simply *group C^*algebra* of G.

(d) $C_r^*(G)$ is called the *reduced group C^*algebra* of G.

Theorem 3.1.8. *[Wiener-Tauberian theorem] Let G be a locally compact group. Then for any $g \in G$, there exists $u \in A(G)$ such that $u(g) \neq 0$.*

As G is a locally compact abelian group, it is known that $\widehat{G} = \{\gamma : G \longrightarrow \mathbb{T} : \gamma$ is a continuous homomorphism$\}$. Under the topology of compact convergence on G and the pointwise product, $\widehat{G}$ becomes a locally compact abelian group. In this case, we call $\widehat{G}$ the *dual group* of G.

Theorem 3.1.9. *Let G be a locally compact abelian group. Then we have the following statements:*

(a) $\sigma(L^1(G)) = \widehat{G}$

(b) $L^1(\widehat{G}) \cong A(G)$

(c) $M(\widehat{G}) \cong B(G)$

(d) $\widehat{\widehat{G}} \cong G$

Therefore, for any locally compact group G, $A(G)$ and $B(G)$ are considered as the "non-abelian" version of $L^1(G)$ and $M(G)$ respectively. Now, we have the following beautiful analogues of corollary 1.1.24 and theorem 1.1.25:

Theorem 3.1.10. *Let G be a locally compact group. Then we have the following statements:*

(a) *G_1 and G_2 are topologically isomorphic if and only if $A(G_1)$ and $A(G_2)$ are isometrically isomorphic.*

(b) *G_1 and G_2 are topologically isomorphic if and only if $B(G_1)$ and $B(G_2)$ are isometrically isomorphic.*

Proof. Refer to [Wal] [Corollary of Theorem 3]. □

3.2 Conditions on $A(G)$ that characterizing the amenability of G

Proposition 3.2.1. *let G be a locally compact group. Then*

(a) *$C_0(G)$ is a Banach $A(G)$-bimodule via the pointwise multiplication.*

(b) *$M(G)$ is a Banach $A(G)$-bimodule via the module actions*

$$d(\mu \cdot g) := d(g \cdot \mu) := gd\mu$$

(c) *$L^1(G)$ is a Banach $A(G)$-submodule of $M(G)$ via the module actions stated in (b).*

Proof.
(a) : For any $f \in C_0(G), g \in A(G)$,

$$\|fg\|_\infty = \|gf\|_\infty \leq \|f\|_\infty \|g\|_\infty \leq \|f\|_\infty \|g\|_{A(G)}$$

(b) : For any $\mu \in M(G), g \in A(G), f \in C_0(G)_{\|\cdot\|_\infty \leq 1}$,

$$|\int_G fd(g \cdot \mu)| = |\int_G fgd\mu| \leq \|fg\|_\infty \|\mu\|_{M(G)} \leq \|g\|_\infty \|\mu\|_{M(G)} \leq \|g\|_{A(G)} \|\mu\|_{M(G)}$$

(c): It trivially follows from (b) and the fact that $L^1(G)$ is a closed ideal in $M(G)$. □

$VN(G)$ is a dual Banach $A(G)$-bimodule in a natural manner. Denote the norm closure of $A(G) \cdot VN(G)$ in $VN(G)$ by $UCB(\widehat{G})$.

Remark. In the view of theorem 1.1.19, when G is abelian, $UCB(\widehat{G})$ is just the space of uniformly continuous functions on $\widehat{G}$.

Theorem 3.2.2. *Let G be a locally compact group. Let $A(G)$ be Fourier algebra. Then the following statements are equivalent.*

(a) *G is amenable.*

(b) *$A(G)$ has a bounded approximate identity in the boundary of its unit circle.*

(c) *$A(G)$ has a bounded approximate identity.*

(d) $C_0(G) = A(G) \cdot C_0(G)$

(e) $M(G) = \mathfrak{M}_{A(G)}(C_0(G), VN(G)) \cong \mathfrak{M}_{A(G)}(A(G), M(G))$

(f) $L^1(G) = \mathfrak{M}_{A(G)}(A(G), L^1(G))$

(g) $B(G) = \mathfrak{M}(A(G), A(G))$

(h) *$A(G)$ is closed in $\mathfrak{M}(A(G), A(G))$.*

(i) *The norm of $A(G)$ is equivalent to that induced by the regular representation of $A(G)$.*

(j) *$A(G)$ is $\sigma(B(G), C^*(G))$ – dense in $B(G)$.*

(k) *If A is a translation-invariant, conjugate invariant subalgebra of $B(G)$ which separates points of G, then A is $\sigma(B(G), C^*(G))$ – dense in $B(G)$.*

(l) *The map $N \longrightarrow B(G/N)$ is a bijection between the set of all closed normal subgroups N of G and the set of all weak*-closed, invariant *-subalgebras A of $B(G)$ with $A \neq 0$.*

(m) $UCB(\widehat{G}) = A(G) \cdot VN(G)$

(n) $A(G) = A(G) \cdot A(G)$

(o) $A(G) = span(A(G) \cdot A(G))$

Step 1: We shall show that "(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a)"

We first have the following lemmas.

Lemma 3.2.3. *Let G be an amenable, locally compact group. Then, for each compact subset K of G and $\varepsilon > 0$, there is a $f \in L^1(G)^+_{\|\cdot\|=1}$ such that $\|\delta_g * f - f\|_1 < \varepsilon \quad (g \in K)$.*

Proof. Fix $f_0 \in L^1(G)^+_{\|\cdot\|=1}$. Since $\lambda_1 : G \longrightarrow \mathfrak{L}(L^1(G)), \lambda_1(g)(f) := g * f \ (f \in L^1(G))$ is a strongly continuous representation, there is a neighborhood U of e_G in G such that

$$\|\delta_g * f_0 - f_0\|_1 = \|\lambda_1(g)f_0 - f_0\|_1 \leq \frac{\varepsilon}{2} \qquad (1)$$

Since G is amenable, there is a net $\{f_\alpha\}_{\alpha\in A}$ in $L^1(G)^+_{\|\cdot\|=1}$ such that $\|\widetilde{f} * f_\alpha - f_\alpha\|_1 \to 0 \quad (\widetilde{f} \in L^1(G)^+_{\|\cdot\|=1})$

and thus, in particular,

$$\|\delta_g * f_0 * f_\alpha - f_\alpha\|_1 \to 0 \quad (g \in G) \qquad (2)$$

Since K is compact, there are $g_1, ..., g_n \in K$ such that $K \subseteq \bigcup_{j=1}^n U g_j$.

By (2), there is $f_{\alpha_0} \in L^1(G)^+_{\|\cdot\|=1}$ such that

$$\|\delta_g * f_0 * f_{\alpha_0} - f_0\|_1 < \frac{\epsilon}{4} \ (j = 1, 2, ..., n) \quad \textit{and} \quad \|f_0 * f_{\alpha_0} - f_0\|_1 < \frac{\epsilon}{4}$$

so that,

$$\|\delta_g * f_0 * f_{\alpha_0} - f_0 * f_{\alpha_0}\|_1 < \frac{\epsilon}{2} \ (j = 1, 2, ..., n)$$

Let $f = f_0 * f_{\alpha_0}$, $h \in K$. Then there is $j \in 1, 2, ..., n$ such that $hg_j \in U$.

We then obtain:

$$\begin{aligned}
&\|\delta_g * f - f\|_1 \\
&\leq \|\delta_{hg_j^{-1}} * \delta_{g_j} * f_0 * f_{\alpha_0} - f_0 * f_{\alpha_0}\|_1 \\
&\leq \|\delta_{hg_j^{-1}} * \delta_{g_j} * f_0 * f_{\alpha_0} - \delta_{hg_j^{-1}} * f_0 * f_{\alpha_0}\|_1 + \|\delta_{hg_j^{-1}} * f_0 * f_{\alpha_0} - f_0 * f_{\alpha_0}\|_1 \\
&= \|\delta_{g_j^{-1}} * f_0 * f_{\alpha_0} - \delta_{hg_j} * f_0 * f_{\alpha_0}\|_1 + \|\delta_{hg_j^{-1}} * f_0 - f_0\|_1 \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \ (g \in K) \qquad \text{by (1) and (2)}
\end{aligned}$$

□

Lemma 3.2.4. *Let $(u_\beta)_\beta$ be a net in the unit ball of $B(G)$ such that $u_\beta \to u_0$ in $\sigma(B(G), C^*(G))$-topology. Let $(E_U)_{U\in\mathbb{B}}$ be the bounded approximate identity of $L^1(G)$ stated in theorem 1.1.18 and let $e_U = E_U * E_U^*$ for any $U \in \mathbb{B}$. Then for any $\epsilon > 0$, there exists U_0 and β_0 such that such that*

$$\|e_{U_0} * u_\beta - u_\beta\|_{B(G)} < \epsilon \ \textit{for any} \ \beta \geq \beta_0$$

and

$$\|e_{U_0} * u_0 - u_0\|_{B(G)} < \epsilon$$

Proof. Assume $u_0 \neq 0$. Let $C^*(G)^\sharp$ be the C*-algebra formed by adjoining $1 := \delta_{e_G}$ to $C^*(G)$. If $f \in L^1(G)$, then since $e_\alpha * = e_\alpha$, for any $u \in B(G)$, we have

$$| < e_\alpha * u - u, f >_{(B(G),C^*(G))} |^2 = | < u, (e_\alpha - 1)^* * f >_{(B(G),C^*(G)^\sharp)} |^2$$

$$\leq \|u\|_{B(G)}| < |u|, (e_\alpha - 1) * f * f^* * (e_\alpha - 1) >_{(B(G),C^*(G)^\sharp)} |$$

$$\leq \|u\|_{B(G)}\|f * f^*\|_{C^*(G)}| < |u|, (1 - e_\alpha)^* * (1 - e_\alpha) >_{(B(G),C^*(G)^\sharp)} |$$

Since $\|e_\alpha\|_{C^*(G)} \leq \|e_\alpha\|_1 \leq 1$, $0 \leq e_\alpha \leq 1$ in $C^*(G)^\sharp$. It follows that

$$0 \leq (1 - e_\alpha)^* * (1 - e_\alpha) \leq 1 - e_\alpha$$

Hence, we have

$$| < e_\alpha * u - u, f >_{(B(G),C^*(G)^\sharp)} |^2 \leq \|u\|_{B(G)}\|f * f^*\|_{C^*(G)}| < |u|, 1 - e_\alpha >_{(B(G),C^*(G)^\sharp)} | \quad (3.1)$$

Note that $|u|(e_\alpha) \to |u|(1) = \|u\|_{B(G)}$. Let α_0 be such that

$$\|u_0\|(< |u_0|, 1 - e_{\alpha_0} >_{(B(G),C^*(G)^\sharp)}) < \epsilon$$

Then

$$\|u_\beta\|(< |u_\beta|, 1 - e_{\alpha_0} >_{(B(G),C^*(G)^\sharp)}) \to \|u_0\|(< |u_0|, 1 - e_\alpha >_{(B(G),C^*(G)^\sharp)}) < \epsilon$$

Choose β_0 such that $\|u_{\beta_0}\|(< |u_{\beta_0}|, 1 - e_{\alpha_0} >_{(B(G),C^*(G)^\sharp)}) < \epsilon$ if $\beta \leq \beta_0$.
By (3.1),

$$\|e_{\alpha_0} * u_\beta - u_\beta\|_{B(G)} \leq \|u_\beta\|_{B(G)}| < |u_\beta|, 1 - e_{\alpha_0} >_{(B(G),C^*(G)^\sharp)} | < \epsilon$$

□

Lemma 3.2.5. *Let $A \subset L^\infty(G)$ be a norm bounded and $f \in L^1(G)$. If $(\phi_\alpha)_\alpha$ is a net in A such that $\phi_\alpha \to \phi$ in $\sigma(L^\infty(G), L^1(G))$-topology, then $f * \phi_\alpha \to f * \phi$ uniformly on compact subsets of G.*

Proof. Fix a compact subset $K \subseteq G$. For any $x \in G$ $f \in L^1(G)$, $G \longrightarrow L^1(G)$, $x \mapsto L_x f$ is continuous, therefore $\{L_x f : x \in K\}$ is a compact subset of $L^1(G)$. Note that on norm bounded subsets, the w*-topology is stronger than the topology of uniform convergence on compacta. Hence,

$$f * \phi_\alpha(x) = < \phi_\alpha, L_x f > \rightarrow < \phi, L_x f > = f * \phi(x)$$

uniformly for $x \in K$. □

Lemma 3.2.6. *Let $U \in B(G)$ and let $(u_\alpha)_\alpha$ be a bounded net in $B(G)$ such that $u_\alpha \to u$ in $\sigma(B(G), C^*(G))$-topology. Then, for any $g, f \in C_c(G)$,*

$$\|((f * g) * u_\alpha)v - ((f * g) * u)v\|_{A(G)} \to 0 \quad (v \in A(G))$$

Proof. Let $\gamma = sup_\alpha \|u_\alpha\|_{B(G)}$. Let $v \in A(G) \cap C_c(G)$ and let K be a compact subset of G such that $(suppf)^{-1}(suppv) \subseteq K$. Then for any $w \in L^\infty(G)$ and $h \in L^1(G)$, we have

$$< (f * w)v, h >_{(L^\infty(G), L^1(G))} = < w, \bar{f} * (vh) >_{(L^\infty(G), L^1(G))}$$

$$= < w\chi_K, \bar{f} * (vh) >_{(L^\infty(G), L^1(G))} = < (f * (w\chi_K))v, h >_{(L^\infty(G), L^1(G))}$$

Note that f, $(w\chi_K)^\sim \in L^2(G)$ which implies that $f * (w\chi_K) \in A(G)$ and

$$\|(f * w)v\|_{A(G)} \|f * (w\chi_K)v\|_{A(G)} \leq \|f\|_2 \|w\chi_K\|_2 \|v\|_{A(G)}$$

Let $w = g * (u_\alpha - u)$. By lemma 3.2.5, we have $g * (u_\alpha - u) \to 0$ uniformly on K. Consequently, we get

$$\|[f*g*(u_\alpha - u)]v\|_{A(G)} \leq \|f\|_2 \|[g*(u_\alpha - u)\chi_K]^\sim\|_2 \|v\|_{A(G)} \to 0 \quad (f, g \in C_c(G), v \in A(G) \cap C_c(G))$$

Since $\|(f*g)*u_\alpha\|_{B(G)} \leq \|f*g\|_{C^*(G)} \|u_\alpha\|_{B(G)} \leq \|f*g\|_{L^1(G)}\gamma$, $((f*g)*u_\alpha)_\alpha$ is a bounded net in $B(G)$. It follows by the density of $A(G) \cap C_c(G)$ in $A(G)$ that

$$\|[f * g * (u_\alpha - u)]v\| \to 0 \quad (f, g \in C_c(G), v \in A(G))$$

□

Lemma 3.2.7. *Let $S = \{u \in B(G) : \|u\|_B = 1\}$. Let $u_\beta \in S$ and $u \in S$. Then the following statements are equivalent:*

(a) $u_\beta \to u$ *in* $\sigma(B(G), C^*(G)) - topology$

(b) $u_\beta \to u$ *uniformly on compact subsets of* G

(c) $u_\beta v \to uv$ *in* $\| \cdot \|_{A(G)}$ $(v \in A(G))$

Proof.
"(a) $\Rightarrow$ (c)" Let $\epsilon > 0$ and U_α be a relatively compact neighborhood of e_G, V_α be a open neighborhood of e_G such that $V_\alpha = V_\alpha^{-1}$ and Let $(e_U)_{U \in \mathbb{B}}$ be the bounded approximate identity of $L^1(G)$ stated in theorem 1.1.18. By lemma 3.2.4, there exists U_0, β_0 such that

$$\|e_{U_0} * u_\beta - u_\beta\|_{B(G)} < \epsilon \ for \ any \ \beta \geq \beta_0$$

and

$$\|e_{U_0} * u - u\|_{B(G)} < \epsilon$$

Take $F = e_{\alpha_0}$ in lemma 3.2.6. Then, there exists $\beta_1 \geq \beta_0$ such that for any $\beta \geq \beta_1$,

$$\|[e_{U_0} * (u_\beta - u)]v\|_{B(G)} \leq \epsilon$$

Thus, if $v \in A(G)$ and $\beta \geq \beta_1$, we have

$$\|(u_\beta - u)v\|_{B(G)} \leq \|(u_\beta - e_{U_0} * \beta)v\|_{B(G)} + \|[e_{U_0} * (u_\beta - u)]v\|_{B(G)} + \|(e_{U_0}) * u - u)v\|_{B(G)} \leq (2\|v\|_{A(G)} + 1)\epsilon$$

"(c) $\Rightarrow$ (b)" Suppose $u_\beta v \to uv$ in $\| \cdot \|_{A(G)}$ $(v \in A(G))$, let K be a compact subset of G. There exists $v \in A(G)$ such that $v|_K \equiv 1$.
Then

$$\|(u_\alpha - u)\chi_K\|_\infty \leq \|(u_\alpha - u)v\|_{A(G)} \to 0$$

"(b) $\Rightarrow$ (a)" By assumption, for any compact subset $K \subseteq G$, $\|(u_\alpha - u)\chi_K\|_\infty \to 0$. Let $f \in C_c(G)$. Then

$$| < u_\alpha - u, f >_{(B(G),C^*(G))} | \leq \int_G |u_\alpha - u||f| = \int_{supp(f)} |u_\alpha - u||f| \leq \|f\|_1 \|(u_\alpha - u)\chi_{supp(f)}\|_\infty \to 0$$

Note that $sup_\alpha \|u_\alpha - u\| \leq 2$ and $C_c(G)$ is dense in $C^*(G)$,

$$| < u_\alpha - u, f >_{(B(G),C^*(G))} | \to 0 \quad (f \in C^*(G)$$

□

Lemma 3.2.8. *Let G be a locally compact group. Then*

$$\mathfrak{M}_{A(G)}(C_0(G), VN(G)) \cong \mathfrak{M}_{A(G)}(A(G), M(G))$$

Proof. For any $\Phi \in \mathfrak{M}_{A(G)}(C_0(G), VN(G))$, define $\Gamma_\Phi \in \mathfrak{L}(A(G), M(G))$ by

$$< f, \Gamma_\Phi(v) >_{(C_0(G),M(G)))} := < \Phi(f), v >_{(VN(G),A(G))} \quad (f \in C_0(G),\ v \in A(G))$$

Since for any $f \in C_0(G), v, g \in A(G))$,

$$< f, \Gamma_\Phi(g \cdot v) > = < \Phi(f), g \cdot v > = < g \cdot \Phi(f), v > = < \Phi(g \cdot f), v > = < g \cdot f, \Gamma_\Phi(v) >$$

We have $\Gamma_\Phi \in \mathfrak{M}_{A(G)}(A(G), M(G))$. Thus,

$$\begin{aligned}
&\|\Gamma_\Phi\|_{\mathfrak{L}(A(G),M(G))} \\
=\ & sup\ \{\|\Gamma_\Phi(v)\|_{M(G)} : v \in A(G)_{\|\cdot\| \leq 1}\} \\
=\ & sup\ \{| < f, \Gamma_\Phi(v) >_{(C_0(G),M(G)))} | : v \in A(G)_{\|\cdot\| \leq 1}, f \in C_0(G)_{\|\cdot\| \leq 1}\} \\
=\ & sup\ \{| < \Phi(f), v) >_{(VN(G),M(G))} | : v \in A(G)_{\|\cdot\| \leq 1}, f \in C_0(G)_{\|\cdot\| \leq 1}\} \\
=\ & sup\ \{\|\Phi(f)\|_{VN(G)} : f \in C_0(G)_{\|\cdot\| \leq 1}\} \\
=\ & \|\Phi\|_{\mathfrak{L}(C_0(G),VN(G))}
\end{aligned}$$

Therefore, $\Phi \mapsto \Gamma_\Phi$ is an isometry.

Conversely, for any $\Gamma \in \mathfrak{M}_{A(G)}(A(G), M(G))$, define $\Phi_\Gamma \in \mathfrak{L}(C_0(G), VN(G))$ by

$$< \Phi_\Gamma(f), v >_{(VN(G),A(G))} := < f, \Gamma(v) >_{C_0(G),M(G))} \quad (f \in C_0(G),\ v \in A(G))$$

Similarly, $\Phi_\Gamma \in \mathfrak{M}_{A(G)}(C_0(G), VN(G))$.

It is easily seen that $\Phi \mapsto \Gamma_\Phi$ and $\Gamma \mapsto \Phi_\Gamma$ are mutually inverse. Thus, $\Phi \mapsto \Gamma_\Phi$ is an isometric isomorphism. □

Remark. For any $f \in L^1(G)$, $u \in A(G)$, $g \in C_0(G)$,

$$\lambda_2(f)(ug) = f * ug = \lambda_2(f)(ug) = [u \cdot \lambda_2(f)](g)$$

So, for any $f \in L^1(G)$, $\lambda_2(f) \in \mathfrak{L}(A(G), L^1(G))$ and

$$\|\lambda_2(f)\|_{\mathfrak{L}(A(G),L^1(G))} = sup\{\|u \cdot \lambda_2(f)\| : u \in A(G), \|u\|_\infty \leq 1\}$$

Lemma 3.2.9. *Let G be a locally compact group. Let $X := \{h \in C_0(G) : h(x) = \sum_{i=1}^\infty u_i(x)g_i(x), u_i \in A(G), g_i \in C_0(G), \sum_{i=1}^\infty \|u_i\|_{A(G)}\|g\|_\infty \leq \infty\}$. Define the norm $\|h\|_X := inf\{\sum_{i=1}^\infty \|u_i\|_{A(G)}\|g_i\|_\infty : h(x) = \sum_{i=1}^\infty u_i(x)g_i(x), u_i \in A(G), g_i \in C_0(G)\}$ $(h \in X)$. Then $\mathfrak{M}_{A(G)}(C_0(G), VN(G)) \cong X^*$ and the duality is given by*

$$< \Phi, h >= \sum_{i=1}^\infty < \Phi(g_i), u_i >_{(VN(G),A(G))}$$

$$(h \in X,\ h(x) = \sum_{i=1}^\infty u_i(x)g_i(x), u_i \in A(G), g_i \in C_0(G),\ \Phi \in \mathfrak{M}_{A(G)}(C_0(G), VN(G)))$$

Proof. Write $\mathfrak{M} := \mathfrak{M}_{A(G)}(C_0(G), VN(G)))$ for convenience.

Let $F \in X^*$. For any $u \in A(G)$, $f \in C_0(G)$, define $\Phi_F \in \mathfrak{M}$ by

$$< \Phi_F(f), u >_{(VN(G),A(G))}:= F(uf)$$

Since

$$| < \Phi_F(f), u >_{(VN(G),A(G))} | \leq |F(uf)| \leq \|F\|_{X^*}\|uf\|_X \leq \|F\|_{X^*}\|u\|_{A(G)}\|f\|_\infty$$

, we have $\|\Phi_F\|_{\mathfrak{L}(C_0(G),VN(G))} \leq \|F\|_{X^*}$

Also, for any $u, v \in A(G)$, $f \in C_0(G)$,

$$< \Phi_F(v \cdot f), u >_{(VN(G),A(G))}= F(u(vf)) = F((uv)f)$$

$$=< \Phi_F(f), uv >_{(VN(G),A(G))} = < v \cdot \Phi_F(f), u >_{(VN(G),A(G))}$$

which implies that $\Phi_F \in \mathfrak{M}$. Hence Φ_F is well-defined.

Conversely, let $\Phi \in \mathfrak{M}$. For any $h \in X$, $h = \sum_{i=1}^{\infty} u_i g_i$, $u_i \in A(G)$, $g_i \in C_0(G)$, define

$$F_\Phi(h) := \sum_{i=1}^{\infty} < \Phi(g_i), u_i >_{(VN(G),A(G))}$$

We have to show that $\Phi(h)$ does not depend on the representation of h.

Let $\{e_\alpha\}_{\alpha \in A}$ be an approximate identity of $C_0(G)$ such that $\{e_\alpha\}_{\alpha \in A} \subseteq A(G)$ and $sup_{\alpha \in A} \|e_\alpha\|_\infty \leq 1$. Let $\epsilon > 0$ and choose N such that

$$\sum_{i=1}^{\infty} \|u_i\|_{A(G)} \|g_i\|_\infty < \epsilon$$

and choose $\alpha_0 \in A$ such that

$$\sum_{i=1}^{\infty} \|u_i\|_{A(G)} \|g_i e_{\alpha_0} - g_i\|_\infty < \epsilon$$

Then

$|F_\Phi(h)|$
$= | \sum_{i=1}^{\infty} < \Phi(g_i), u_i >_{(VN(G),A(G))} |$
$\leq | \sum_{i=1}^{\infty} < \Phi(g_i e_{\alpha_0} - g_i), u_i >_{(VN(G),A(G))} | + | \sum_{i=1}^{\infty} < \Phi(g_i e_{\alpha_0}), u_i >_{(VN(G),A(G))} |$
$\leq \sum_{i=1}^{\infty} \|\Phi(g_i e_{\alpha_0} - g_i)\|_{VN(G)} \|u_i\|_{A(G)} + | \sum_{i=1}^{\infty} < e_{\alpha_0} \cdot \Phi(g_i), u_i >_{(VN(G),A(G))} |$
$\leq \|\Phi\|_{\mathfrak{M}} \sum_{i=1}^{\infty} \|g_i e_{\alpha_0} - g_i)\|_\infty \|u_i\|_{A(G)} + | \sum_{i=1}^{\infty} < \Phi(g_i), e_{\alpha_0} u_i >_{(VN(G),A(G))} |$
$\leq \|\Phi\|_{\mathfrak{M}} (\epsilon + 2\epsilon) + | \sum_{i=1}^{\infty} < g_i, \Gamma_\Phi(e_{\alpha_0} u_i) >_{(C_0(G),M(G))} |$
$\leq 3\epsilon \|\Phi\|_{\mathfrak{M}} + | \sum_{i=1}^{\infty} < g_i, u_i \cdot \Gamma_\Phi(e_{\alpha_0}) >_{(C_0(G),M(G))} |$
$\leq 3\epsilon \|\Phi\|_{\mathfrak{M}} + | \sum_{i=1}^{\infty} < u_i g_i, \Gamma_\Phi(e_{\alpha_0}) >_{(C_0(G),M(G))} |$
$\leq 3\epsilon \|\Phi\|_{\mathfrak{M}} + | < h, \Gamma_\Phi(e_{\alpha_0}) >_{(C_0(G),M(G))} |$

Therefore, if $h = 0$, then $F(h) = 0$. Hence, the claim is proven.

Let $\|h\|_X \leq 1$, $\epsilon > 0$. Choose $g_i \in C_0(G)$, $u_i \in A(G)$ such that $h = \sum_{i=1}^{\infty} \|u_1\|_{A(G)} \|g_i\|_\infty \leq \|h\|_X + \epsilon$.

Then $|F(h)| \leq \|\Phi\|_{\mathfrak{M}} \sum_{i=1}^{\infty} \|g_i)\|_\infty \|u_i\|_{A(G)} = \|\Phi\|_{\mathfrak{M}} (1 + \epsilon)$, and so $\|F\|_{X^*} \leq \|\Phi\|_{\mathfrak{M}}$. $\square$

We may now continue "**Step 1**".

Proof.
"(a) $\Rightarrow$ (b)" Let $\{f_\alpha\}_{\alpha\in A}$ be a net in $L^1(G)^+_{\|\cdot\|=1}$ such that $\|\delta_g * f_\alpha - f_\alpha\|_1 \to 0$ uniformly on compact subsets of G. Let $\xi_\alpha := (f_\alpha)^{\frac{1}{2}}$, and $e_\alpha := \xi_\alpha * \check{\xi_\alpha}$.

We claim that: $\{e_\alpha\}_{\alpha\in A}$ is a net in $A(G)_{\|\cdot\|=1} \subseteq B(G)_{\|\cdot\|=1}$ and $e_\alpha \to 1$ uniformly on compact subsets of G.

"**Proof of the claim**:" Note that

$$\|e_\alpha\|_{A(G)} = \|\xi_\alpha * \check{\xi_\alpha}\|_{A(G)} \leq (\|\xi_\alpha\|_2)^2 = (\|f_\alpha\|_1) = 1$$

and

$$\|e_\alpha\|_{A(G)} = sup\{| < T\xi_\alpha, \xi_\alpha >: T \in VN(G)\} \geq | < \lambda_2(e_G)\xi_\alpha, \xi_\alpha > | = | < \xi_\alpha, \xi_\alpha > | = \|f_\alpha\|_1 = 1$$

The first statement follows. For the second statement,

$$\begin{aligned}
&|\xi_\alpha * \check{\xi_\alpha}(x) - 1| \\
&= |\textstyle\int_G f_\alpha(xy)^{\frac{1}{2}} f_\alpha(y)^{\frac{1}{2}} dy - 1| \\
&= |\textstyle\int_G f_\alpha(xy)^{\frac{1}{2}} f_\alpha(y)^{\frac{1}{2}} dy - \int_G f_\alpha(xy) dy| \\
&\leq \textstyle\int_G |f_\alpha(xy)^{\frac{1}{2}}||f_\alpha(y)^{\frac{1}{2}} - f_\alpha(xy)^{\frac{1}{2}}| dy \\
&\leq \|\delta_x * f_\alpha^{\frac{1}{2}}\|_2 \|f_\alpha^{\frac{1}{2}} - (\delta_x * f_\alpha)^{\frac{1}{2}}\|_2 \\
&\leq 1 \cdot \|f_\alpha - \delta_x * f_\alpha\|_1^{\frac{1}{2}} \to 0 \text{ uniformly on compact subsets of } G.
\end{aligned}$$

This, however, is already sufficient for $\{e_\alpha\}_\alpha$ to be a bounded approximate identity for $A(G)$ by lemma 3.2.7.

"(b) $\Rightarrow$ (c)" Trivial.

"(c) $\Rightarrow$ (d)" By Cohen's factorization theorem, the result follows.

"(d) $\Rightarrow$ (e)" It follows from lemma 3.2.9.

"(e) $\Rightarrow$ (f)" Note that $\mathfrak{M}_{A(G)}(A(G), L^1(G)) \subseteq \mathfrak{M}_{A(G)}(A(G), M(G)) = M(G)$. For any $\Gamma \in \mathfrak{M}_{A(G)}(A(G), L^1(G))$, there exists $\mu \in M(G)$ such that $\Gamma(u) = u\mu$ and $u\mu \in L^1(G)$ $(u \in A(G))$. Let E be a Borel set in G such that $m_G(E) = 0$. Then

$$\int_E d(u \cdot \mu)(x) = \int_E u(x) d\mu(x) = 0 \ \ (u \in A(G))$$

For any compact subset $K \subseteq G$, there exists $f \in A(G)$ such that $0 \leq f \leq 1$ and $f = 1$ on K. Thus

$$\mu(K \cap E) = \int_E \chi_K(x) d\mu(x) \leq \int_E u(x) d\mu(x) = 0$$

By regularity of μ, we have $\mu(E) = 0$. As a result, μ is absolutely continuous with respect to m_G, and so $\mu \in L^1(G)$.

"(f) $\Rightarrow$ (a)" If $g \in L^2(G)$, $f \in L^1(G)$, $u \in A(G)$,

$$\|\lambda_2(uf)g\|_2 = \|uf * g\|_2 \leq \||uf| * |g|\|_2 \leq \|\|u\|_\infty |f| * |g|\|_2 \leq \|u\|_\infty \|\lambda_2(|f|)\| \|g\|_2$$

Therefore,

$$\|u \cdot \lambda_2(f)\|_{\mathfrak{M}_{A(G)}(A(G), L^1(G))} = \|\lambda_2(uf)\| \leq \|u\|_\infty \|\lambda_2(f)\|$$

If $f \in L^1(G)^+$, then

$$\|\lambda_2(f)\|_{\mathfrak{M}_{A(G)}(A(G), L^1(G))} \leq \|\lambda_2(f)\| \tag{3.2}$$

If (f) holds, the norm $\|\cdot\|_1$ of $L^1(G)$ is equivalent to $\|\cdot\|_{\mathfrak{M}_{A(G)}(A(G), L^1(G))}$. Combining with (3.2), there exists a positive number K such that

$$\|f\|_1 \leq \|\lambda_2(f)\| \quad (f \in L^1(G)^+)$$

By replacing f by $f * f^*$, we have

$$(\int_G f(x))^2 \leq M \|\lambda_2(f)\|^2$$

By induction,

$$(\int_G f(x)) \leq M^{1/2n}\|\lambda_2(f)\| \;\; \forall n \in \mathbb{N}$$

Therefore, we have $|\int_G f(x)| \leq \|\lambda_2(f)\|$. By theorem 2.1.5, G is amenable.

□

Remark. Lemma 3.2.3 is well-known and can be founded in [Run] [Lemma 7.11] for example. The remaining theorems and lemmas in the above are from [Fig], [Gra-Lj, [Neb] and [Lau].

Step 2: We shall show that "(a) ⇒ (g) ⇒ (h) ⇒ (i) ⇒ (a)"

Proof.
"(a) ⇒ (g)" Let $T \in \mathfrak{M}(A(G), A(G))$, and let $\{e_\alpha\}_{\alpha\in A}$ be an approximate identity with $sup_{\alpha\in A}\|e_\alpha\| \leq M$ with $M > 0$. Note that $\{Te_\alpha\}_{\alpha\in A}$ may be regarded as linear functionals on $C^*(G)$, by boundedness of T, $\{Te_\alpha\}_{\alpha\in A}$ has a $\sigma(B(G), C^*(G))$-limit point $w \in B(G)$. Let $\{Te_\beta\}_{\beta\in B}$ be a subnet of $\{Te_\alpha\}_{\alpha\in A}$ such that $Te_\beta \to w$ in $\sigma(B(G), C^*(G)) - topology$. Fix $u \in A(G)$, note that

$$|\int_G (uTe_\beta - uw)(f)| \leq \|u\|_\infty |\int_G (Te_\beta - w)(f)| \to 0 \;\; (f \in L^1(G))$$

Therefore, by density of $L^1(G)$ in $C^*(G)$ and the boundedness of the net $\{uTe_\beta - uw\}_{\beta\in B}$, we get

$$uTe_\beta \to uw \;\; in \;\; \sigma(B(G), C^*(G)) - topology.$$

By boundedness of T again,

$$Tu = lim_\alpha T(ue_\alpha) = lim_\alpha uT(e_\alpha) = uw$$

Finally, we get $T = L_w$ for some $w \in B(G)$ where $L_w : A(G) \longrightarrow A(G), u \mapsto uw$ By theorem 3.1.8, if $uw_1 = uw_2$ $(u \in A(G))$, then $w_1(g) = w_2(g)$ $(g \in G)$. It follows that there exists a unique $w \in B(G)$ such that $T = L_w$. Clearly $\|Tu\| \leq \|u\|\|w\|$ implies that $\|T\| \leq \|w\|$. On the other hand, $Te_\beta \to w$ in $\sigma(B(G), C^*(G)) - topology$ and

$B(G)_{\|\cdot\|\leq\|T\|}$ is $\sigma(B(G), C^*(G))$ – *closed*, we have $\|w\| \leq \|T\|$. Therefore, $B(G) \longrightarrow \mathfrak{M}(A(G), A(G)), w \mapsto L_w$ is an isometry.

"(g) $\Rightarrow$ (h)" This is trivial since $A(G)$ is always a closed ideal in $B(G)$.

"(h) $\Rightarrow$ (i)" If $(A(G), \|\cdot\|)$ is closed in $\mathfrak{M}(A(G), A(G))$, then the norms $\|\cdot\|_{A(G)}$ and $\|\cdot\|_{\mathfrak{M}(A(G),A(G))}$ are equivalent by open mapping theorem.

"(i) $\Rightarrow$ (a)" Refer to [Los2] [Theorem 1].

□

Remark. The proof of "(a) $\Rightarrow$ (g)" is founded in [Ren].

Step 3: We shall show that "(a) $\Longleftrightarrow$ (j) *and* **(a) $\Rightarrow$ (l) $\Rightarrow$ (k)"**

We quote the following lemma before continuing to proof the main theorem for convenience.

Lemma 3.2.10. *Let G be a locally compact group. Let A be a weak*-closed, invariant subalgebra of $B(G)$. Suppose A is conjugation invariant and separates the points of G. Then A contains $A(G)$.*

Proof. Refer to [B-L-S] [Theorem 1.3]. □

We may now proceed "**Step 3**".

Proof.

"(a) $\Rightarrow$ (j)" Let $\{e_\alpha\}_{\alpha\in A}$ be a bounded approximate identity in $A(G)_{\|\cdot\|=1}$. Then

$$e_\alpha w \to w \ \ in \ \|\cdot\|_{A(G)} \ (w \in A(G))$$

By lemma 3.2.7,

$$e_\alpha \to 1 \ \ in \ \sigma(B(G), C^*(G)) - topology$$

For any $v \in B(G)$, note that $(ve_\alpha)_\alpha \subseteq A(G)$, and thus

$$|\int_G (ve_\alpha - v)(f)| \leq \|v\|_\infty |\int_G (e_\alpha - 1)(f)| \to 0 \quad (f \in L^1(G))$$

Since $(ve_\alpha - v)_\alpha$ is a bounded net, it follows that

$$ve_\alpha \to v \text{ in the } \sigma(B(G), C^*(G)) \text{ topology}$$

"(j) $\Rightarrow$ (a)" Since $A(G)$ is $\sigma(B(G), C^*(G))-dense$ in $B(G)$, there is a net $\{e_\alpha\} \subseteq A(G)$ such that $e_\alpha \to 1$ in $\sigma(B(G), C^*(G)) - topology$. By lemma 3.2.7,

$$e_\alpha w \to w \ \ in \ \| \cdot \|_{A(G)} \ (w \in A(G))$$

"(a) $\Rightarrow$ (l)" Suppose G is amenable, and let $A \neq 0$ be a weak*-closed invariant *-subalgebra of $B(G)$. Then $N = \{x \in G : \delta_x * \varphi = \varphi \ \forall \varphi \in A\}$ is a closed normal subgroup of G. Moreover, A viewed as a subalgebra of $B(G/N)$ separates the points of G/N. By lemma 3.2.10 $A(G/N) \subseteq A$, . Since G/N is amenable by theorem?? and "(a) $\iff$ (j)", $A(G/N)$ is weak*-dense in $B(G/N)$ and hence $A = B(G/N)$. Therefore, the map is well-defined and surjective. If $B(G/N) = B(G/N')$, then

$$N = \{x \in G : \delta_x * \varphi = \varphi \ \forall \varphi \in B(G/N)\} = \{x \in G : \delta_x * \varphi = \varphi \ \forall \varphi \in B(G/N')\} = N'$$

Therefore, the map is bijective.

"(l) $\Rightarrow$ (k)" Let A be the $\sigma(B(G), C^*(G)) - closure$ of $A(G)$ in $B(G)$. Note that $A(G)$ is point separating and translation invariant, and so is A. If $A = B(G/N)$ for some non-trivial closed normal subgroup $N \trianglelefteq G$, then A cannot separates points in N. This forces $A = B(G)$.

"(k) $\Rightarrow$ (a)" Note that $A(G)$ is a translation-invariant, conjugate invariant subalgebra of $B(G)$ which separates points of G, so $A \supseteq A(G)$ is $\sigma(B(G), C^*(G)) - dense$ in $B(G)$.

□

Remark. All of the proofs in this step can be founded in [B-L-S].

Step 4: We shall show that "(a) $\iff$ (m)"

We have the following lemma which is essential in this step. It is just [Lau] [Proposition 4.4].

Lemma 3.2.11. *Let G be a locally compact group. then $UCB(\widehat{G})$ is a subspace of $VN(G)$ containing $C_r^*(G)$.*

Proof. Suppose $f \in C_c(G)$,and *supp*$(f) \subseteq K$ for some compact subset $K \subseteq G$. Let $\phi \in A(G)$ such that $\phi(t) = 1$ for all $t \in K$. Then $\phi \cdot \lambda_2(f) = \lambda_2(f)$. Therefore, $\lambda_2(L^1(G)) \subseteq UCB(\widehat{G})$ which gives $C_r^*(G) \subseteq UCB(\widehat{G})$. □

We may now proceed "**Step 4**".

Proof.
"(a) $\Rightarrow$ (m)" Since "(a) $\iff$ (b)", $A(G)$ has a bounded approximate identity, by Cohen's factorization theorem, the result follows.

"(m) $\Rightarrow$ (a)" Define

$$j' : A(G) \times VN(G) \longrightarrow UCB(\widehat{G}),\ (u, T) \mapsto u \cdot T$$

This induces a bounded linear mapping $j : A(G) \otimes VN(G) \longrightarrow UCB(\widehat{G})$.
Let $j^* : UCB(\widehat{G})^* \longrightarrow (A(G) \otimes VN(G))^* \cong \mathfrak{L}(A(G), VN(\widehat{G})^*)$ be its Banach adjoint. By assumption, j is onto. It follows that there exists a real number $c > 0$ such that $\|j^*(f)\| \geq c\|f\|$ for all $f \in UCB(\widehat{G})^*$. It is not hard to see that $j^*(UCB(\widehat{G})^*) \supseteq \mathfrak{M}(A(G), A(G))$. Recall that $A(G)^* \cong VN(G)$ and $UCB(\widehat{G}) \subseteq VN(G)$. Define $i : A(G) \longrightarrow UCB(\widehat{G})^*$ be the restriction map. Since $C_r^*(G)$ is σ-weakly dense in $VN(G)$, $C_r^*(G)_{\|\cdot\|\leq 1}$ is σ-weakly dense in $VN(G)_{\|\cdot\|\leq 1}$ by Kaplansky density theorem. By lemma 3.2.11, we have $UCB(\widehat{G}) \supseteq C_r^*(G)$, which implies that $i : A(G) \longrightarrow UCB(\widehat{G})^*$ is an isometry. Now

$$j^* \circ i : A(G) \longrightarrow \mathfrak{M}(A(G), A(G)) \subseteq \mathfrak{L}(A(G), VN(\widehat{G})^*)$$

is just the canonical embedding. Therefore, $(j^* \circ i)(A(G))$ is closed in $\mathfrak{M}(A(G), A(G))$. Since "(a) $\iff$ (f)", G is amenable.

$\square$

Remark. "(a) $\Rightarrow$ (m)" is just [Gra] [Proposition 1] and "(m) $\Rightarrow$ (a)" is referred to [L-L] [Proposition 7.1].

Step 5: We shall show that "(a) $\Rightarrow$ (n) $\Rightarrow$ (o) $\Rightarrow$ (a)"

There are several lemmas before finishing the proof of the main theorem which can be founded in [F-G-L].

Definition 3.2.12. Let B be a Banach algebra of complex-valued continuous functions on a topological space. B is said to be *weakly self-adjoint* is there exists a $K_0 > 0$ such that

$$|f|^2 \in B \;\; and \;\; \||f|\|_B \leq K_0 \|f\|_B^2 \quad (f \in B)$$

Lemma 3.2.13. *If B is a Banach algebra of complex-valued continuous functions on a topological space with weak factorization, then there exist N, $K \in \mathbb{N}$ such that*

$$B(N, K) := \{f \in B : f = \sum_{i=1}^{N} \prod_{j=1}^{4} f_i^j \;\; with \;\; \sum_{i=1}^{N} \prod_{j=1}^{4} \|f_i^j\| \leq K\|f\|\}$$

is dense in B.

Proof. Clearly, $B = \bigcup_{N=1}^{\infty} \bigcup_{K=1}^{\infty} B(N, K)$. By the Baire category theorem, there exist $N', K' \in \mathbb{N}$ such that $\overline{B(N', K')}^{\circ} \neq \phi$. Thus there exists $f_0 \in B(N', K')$, $\delta > 0$ such that $B(N', K') \supseteq \{f \in B : \|f - f_0\| < \delta\}$.

Write $f_0 = \sum_{i=1+N'}^{2N'} \prod_{j=1}^{4} f'^j_i$ with $\sum_{i=1+N'}^{2N'} \prod_{j=1}^{4} \|f_i^j\| \leq K'\|f_0\|$ where $(f_i^j)_{i=1+N'}^{2N'} \subseteq B$, $j = 1, 2, 3, 4$.

Let $\omega = e^{\frac{\pi i}{4}}$ and $f_i^j = \omega f'^j_i$, then

$$-f_0 = \sum_{i=1+N'}^{2N'} \prod_{j=1}^{4} f_i^j$$

and

$$\sum_{i=1+N'}^{2N'}\prod_{j=1}^{4}\|f'^j_i\| = \sum_{i=1+N'}^{2N'}\prod_{j=1}^{4}\|f^j_i\| \leq K'\|f_0\|$$

Let $f \in B$ be such that $\delta/2 \leq \|f\| \leq \delta$. Then $\|(f+f_0) - f_0\| < \delta$ implies that $f + f_0 \in \overline{B(N', K')}$. For any $\epsilon \in [0, \delta/4)$, there exist $(f^j_i)_{i=1}^{N'} \subseteq B$, $j = 1, 2, 3, 4$ such that $\|(f+f_0) - \sum_{i=1}^{N'}\prod_{j=1}^{4} f^j_i\| < \epsilon$ and

$$\sum_{i=1}^{N'}\prod_{j=1}^{4}\|f^j_i\| \leq K'(\|f\| + \|f_0\| + \epsilon) \leq K'(\delta + \|f_0\| + \epsilon) \leq K'(2\delta + \|f_0\|)$$

Now, we obtain

$$\|f - \sum_{i=1}^{2N'}\prod_{j=1}^{4} f^j_i\| = \|f - (-f_0 + \sum_{i=1}^{N'}\prod_{j=1}^{4} f^j_i)\| < \epsilon$$

,

$$\sum_{i=1}^{2N'}\prod_{j=1}^{4}\|f^j_i\| \leq K'(2\delta + \|f_0\|) + K'\|f_0\| = 2K'(\|f_0\| + \delta)$$

and

$$\|\sum_{i=1}^{2N'}\prod_{j=1}^{4} f^j_i\| \geq \delta/2 - \epsilon \geq \delta/4$$

We set $N = 2N'$ and $K = 8K'(\|f_0\| + \delta)/\delta$, then $\sum_{i=1}^{2N'}\prod_{j=1}^{4}\|f^j_i\| \leq K\|\sum_{i=1}^{2N'}\prod_{j=1}^{4} f^j_i\|$. Thus, $f \in \overline{B(N, K)}$. Since $tB(N,K) = B(N,K)$ for all $t > 0$. We see that $B = \overline{B(N,K)}$.

□

Lemma 3.2.14. *Let B be a weakly self-adjoint Banach algebra of complex-valued continuous functions with weak factorization. Let $s(B) := \{x : f(x) \neq 0 \text{ for some } f \in B\}$. Then there exists $K_1 > 0$ such that for each compact subset $M \subseteq s(B)$ and any $f_1 \in B$ such that $|f| > 0$ on M, there exists $f_2 \in B$ such that*

$$f_2 \geq |f_1|^{1/2} \text{ and } \|f_2\| \leq K_1\|f_1\|^{1/2}$$

Proof. By lemma 3.2.13, there exist $N, K \in \mathbb{N}$ such that $B(N, K) = B$. Thus, there is a $f'_1 \in B(N, K)$ such that

$$\|f_1 - f'_1\| \leq \frac{1}{4}\,\mathrm{inf}_{x\in M}|f_1(x)|$$

Write $f_1' = \sum_{i=1}^N \prod_{j=1}^4 f_i^j$ with $\sum_{i=1}^N \prod_{j=1}^4 \|f_i^j\| \leq K\|f_1'\|$. Without loss of generality, we may assume that

$$\|f_i^1\| = ... = \|f_i^4\| \leq (K\|f_1'\|)^{1/4} \leq (2K\|f_1\|)^{1/4}$$

Set $f_2' = 1/4\sum_{i=1}^N \prod_{j=1}^4 |f_i^j|^2$, so $f - 2' \leq 0$. By the weak self-adjointness, we have $f_2' \in B$ and

$$\|f_2'\| \leq (2K)^{1/2} N K_0 \|f_1\|^{1/2}$$

Note that

$$f_2' \geq \sum_{i=1}^N \prod_{j=1}^4 |f_i^j|^{1/2} \geq (\sum_{i=1}^N \prod_{j=1}^4 |f_i^j|)^{1/2} \geq |\sum_{i=1}^N \prod_{j=1}^4 f_i^j|^{1/2} = |f_1'|^{1/2}$$

Now $|f_1(x) - f_1'(x)| \leq \|f_1 - f_1'\| \leq \frac{1}{4}|f_i(x)|$ for all $x \in M$, so $|f_2| \geq |f_1'|^{1/2} \geq (\frac{3}{4})^{1/2}|f_1|^{1/2}$. Take $f_2 = \frac{4}{3}f_2'$, $K_1 = 2K^{1/2}NK_0$. Then $f_2 \geq |f_1|^{1/2}$ and

$$\|f_2\| \leq 4/3(2K)^{1/2} N K_0 \|f_1\|^{1/2} \leq K_1 \|f_1\|^{1/2}$$

□

Lemma 3.2.15. *Let B be a weakly self-adjoint Banach algebra of complex-valued continuous functions, and let $s(B) := \{x : f(x) \neq 0 \ for \ some \ f \in B\}$ Suppose that B has weak factorization. Then there exists $K_1 > 0$ such that for each compact subset $M \subseteq s(B)$ there is an element $f \in B$ such that $f \geq 1$ on M, $f \geq 0$ on $s(B)$, and $\|f\| \leq K_1$.*

Proof. By assumptions on B and M, there exists $f_1 \in B$ such that $f_1 \geq 1$ on M. By lemma 3.2.14, there are $f_2, f_3 ... \in B$ such that $|f_{j+1}(x)| \geq |f_j(x)|^{1/2}$ for any $x \in M$ and $\|f_{j+1}\| \geq K_1\|f_j\|^{1/2}$. Hence, $f_{j+1} > 1$ on M and $\|f_{j+1}\| \leq K_1^{\sum_{i=0}^j 2^{-i}} \|f_1\| 2^{-j}$. Choose $K_2 = 2K_1^2$ and $j \in N$ such that $\|f_j\| 2^{-j} \leq 2$. Then $f = f_{j+1}$ will be the required $f \in B$. □

We may now proceed "**Step 5**".

Proof.

"(a) $\Rightarrow$ (n)" Since "(a) $\iff$ (b)", $A(G)$ has a bounded approximate identity, by Cohen's factorization theorem, the result follows.

"(n) $\Rightarrow$ (o)" Trivial.

"(o) $\Rightarrow$ (a)"

We first claim that: $|\int_G g(x)dm_G(x)| \leq \|\lambda_2(g)\| \quad (g \in L^1(G)^+)$

"**Proof of the claim**:" By lemma 3.2.15, there is $M > 0$ such that for each compact subset $K \subseteq G$ there is an element $f \in A(G)$ such that $f \geq 1$ on K, $f \geq 0$ on G, and $\|f\|_{A(G)} \leq M$.

Clearly, for any $g \in L^1(G)^+$, $\|\lambda_2(g)\|$ is equal to the norm of the linear functional $h \mapsto \int_G h(x)g(x)$ on $A(G)$. Therefore,

$$|\int_G f(x)g(x)dm_G(x)| \leq \|\lambda_2(g)\|\|f\|_{A(G)}$$

$$\Rightarrow \int_K g(x)dm_G(x) \leq |\int_G f(x)g(x)dm_G(x)| \leq M\|\lambda_2(g)\|$$

By regularity of Haar measure,

$$\int_G g(x)dm_G(x) \leq M\|\lambda_2(g)\|$$

By replacing g by $g * g^*$, we have

$$(\int_G g(x))^2 \leq M\|\lambda_2(g)\|^2$$

We hence obtain by induction that,

$$(\int_G g(x)) \leq M^{1/2n}\|\lambda_2(g)\| \text{ for all } n \in \mathbb{N}$$

Therefore, we have $|\int_G g(x)| \leq \|\lambda_2(g)\|$. By theorem 2.1.5, G is amenable. $\square$

Remarks. 1. The proof of "(o) $\Rightarrow$ (a)" is a blend of [Los1] [Proposition 2] and [Her] [Lemma 4].

2. There are only a few groups having amenable Fourier algebras. Let G be a locally compact group and $A(G)$ be its Fourier algebra. Then $A(G)$ is amenable if and only if G has an abelian group of finite index [F-R]. Therefore, amenability is a rather restrictive requirement for Fourier algebras.

Chapter 4

Operator Amenability and Fourier algebras

4.1 Operator Amenability

Before the discussion of the main theorems, we need some preliminary. Those results presented below can be found in [Rua1] unless specified.

Definition 4.1.1. A *matrix norm* $\|\cdot\|$ on a vector space V is an assignment of a norm $\|\cdot\|_n$ on the matrix space $\mathbb{M}_n(V)$ for each $n \in \mathbb{N}$. An *operator space* is a vector space together with a matrix norm $\|\cdot\|$ for which

1. $\|v \oplus w\|_{m+n} = \max(\|v\|_n, \|w\|_m)$
2. $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_m \|\beta\|$

for all $v \in \mathbb{M}_m(V)$, $w \in \mathbb{M}_n(V)$, $\alpha \in M_{n,m}$, $\beta \in M_{m,n}$, $m, n \in \mathbb{N}$.

Definition 4.1.2. Given two operator spaces V and W and a linear mapping $\phi : V \longrightarrow W$, for each $n \in \mathbb{N}$, there is a corresponding linear mapping $\phi_n : \mathbb{M}_n(V) \longrightarrow \mathbb{M}_n(W)$ defined by

$$\phi_n(v) := [\phi(v_{i,j})] \quad (v = [v_{i,j}] \in \mathbb{M}_n(V))$$

We define the *complete bounded norm* of ϕ by

$$\|\phi\|_{cb} := \sup\{\|\phi_n\| : n \in \mathbb{N}\}$$

We say that

- ϕ is *completely bounded* if $\|\phi\|_{cb} < \infty$
- ϕ is *completely contractive* if $\|\phi\|_{cb} \leq 1$
- ϕ is a *completely isometry* if each ϕ_n is an isometry.
- V and W are completely isometrically isomorphic if there is a completely isometrically isomorphism from V onto W. In this case, we write $V \cong_{c.i.} W$

Denote by $\mathcal{CB}(V, W)$ the vector space of all completely bounded linear maps from V to W.

Theorem 4.1.3. *For each Hilbert space $\mathcal{H}$, every subspace W of $\mathfrak{L}(\mathcal{H})$ is a operator space where the matrix norm is inherited from $M_n(\mathfrak{L}(\mathcal{H})) = \mathfrak{L}(\mathcal{H}^n)$.*
Conversely, if V is a operator space, then there exists a Hilbert space H, a concrete operator space $W \subseteq \mathfrak{L}(\mathcal{H})$, and a complete isometry Φ of V onto W.

Theorem 4.1.4 (Arveson-Wittstock extension theorem). *If V is a subspace of an operator space W, and $\mathcal{H}$ is a Hilbert space, then any complete contraction $\phi : V \longrightarrow \mathfrak{L}(\mathcal{H})$ has a completely contractive extension $\Phi : W \longrightarrow \mathfrak{L}(\mathcal{H})$.*

Definition 4.1.5. An *operator space system* V on a Hilbert space $\mathcal{H}$ is a norm closed linear subspace $V \subseteq \mathfrak{L}(\mathcal{H})$ which is self-adjoint, i.e. $x \in V$ if and only if $x \in V^*$, and unital.

Definition 4.1.6. Given operator systems V and W, a linear mapping $\phi : V \longrightarrow W$ is said to be *completely positive* if $\phi_n \geq 0$ for all $n \in \mathbb{N}$.

Theorem 4.1.7. *If $\phi : V \longrightarrow W$ is a linear mapping of operator systems such that $\phi(I) = I$, then ϕ is completely positive if and only if ϕ is a complete contraction.*

Proposition 4.1.8. *Let V, W be two operator spaces. For any $v \in M_n(V \otimes W)$, we define*

$$\|v\|_{op} := inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta, v \in M_p(V), w \in M_q(W), \alpha \in M_{n,p\times q}, \beta \in M_{p\times q,n}\}$$

Then $(V \otimes W, \|\cdot\|_{op})$ is an operator space and we define the operator space projective tensor product $V\widehat{\widehat{\otimes}}W$ to be the completion of this space.

Definition 4.1.9. Let A be an operator space. A bilinear map $m : A \times A \longrightarrow A$ is said to be *completely contractive* if it determines a completely contractive linear map

$$m : A\widehat{\widehat{\otimes}}A \longrightarrow A$$

$$\text{i.e. } \|[a_{i,j}b_{k,l}]\| \leq \|a\|\|b\| \quad (a = [a_{i,j}] \in M_m(A)\ , b = [b_{k,l}] \in M_n(A)\ , m, n \in \mathbb{N})$$

Proposition 4.1.10. *Given two operator spaces V, W, then*

$$(V\widehat{\widehat{\otimes}}W)^* \cong_{c.i.} \mathcal{CB}(V, W^*)$$

Definition 4.1.11. Let A be a Banach algebra which is also an operator space. A is said to be *completely contractive* if the multiplication of it is a completely contractive bilinear map.

Definition 4.1.12. Let A be a completely contractive Banach algebra. An *operator A − bimodule* E is an A-bimodule which is also an operator space such that the module actions

$$A \times E \longrightarrow E,\ (a, x) \mapsto a \cdot x \text{ and } E \times A \longrightarrow E,\ (x, a) \mapsto x \cdot a$$

are completely bounded.

Proposition 4.1.13. *Let A be a completely contractive Banach algebra and let X be an operator A-bimodule. Denote by $A^\sharp$ the unitization of A. Then X is a operator $A^\sharp$-bimodule by setting*

$$1 \cdot x = x \text{ and } x \cdot 1 = x \quad (x \in A)$$

Moreover, $A\widehat{\widehat{\otimes}}X$ is completely isometrically imbedded into $A^\sharp\widehat{\widehat{\otimes}}X$.

Proof. Note that $A^\sharp\widehat{\widehat{\otimes}}X \cong (A\widehat{\widehat{\otimes}}X)\oplus_1 X$ and the left module multiplication $m_l : A^\sharp\widehat{\widehat{\otimes}}X \longrightarrow X$ corresponds to the complete contraction

$$(A\widehat{\widehat{\otimes}}X) \oplus_1 X \longrightarrow X,\ (a \otimes x) \oplus y \mapsto a \cdot x + y \ \ (a \in A, x, y \in X)$$

The case of the right module multiplication is similar.

Let $\pi : A^\sharp \longrightarrow A$ be the complete quotient map and $i : A \longrightarrow A^\sharp$ the natural injection. Then $\mathfrak{id}_{A\widehat{\widehat{\otimes}}X}$ is decomposed into

$$A\widehat{\widehat{\otimes}}X \xrightarrow{i\otimes\mathfrak{id}_X} A^\sharp\widehat{\widehat{\otimes}}X \xrightarrow{\pi\otimes\mathfrak{id}_X} A\widehat{\widehat{\otimes}}X$$

Hence, for any $u \in \mathbb{M}_n(A\widehat{\widehat{\otimes}}X)$,

$$\|u\| \leq \|\pi \otimes \mathfrak{id}_X\|\|(i \otimes \mathfrak{id}_X)u\| \leq \|u\|$$

$$\Rightarrow \|(i \otimes \mathfrak{id}_X)u\| = \|u\|$$

□

Definition 4.1.14. Let A be a completely contractive Banach algebra. If E is a operator A-bimodule, then is it not hard to show that E^* becomes a operator A-bimodule through

$$< x, \phi \cdot a >:=< a \cdot x, \phi >,\ \ < x, a \cdot \phi >:=< x \cdot a, \phi > \ \ (a \in A, x \in E, \phi \in E^*)$$

The dual space of an operator bimodule equipped with these module operations is called *dual operator bimodule.*

Definition 4.1.15. A completely contractive Banach algebra A is said to be *operator-amenable* if for every dual operator A-bimodule E, every completely bounded derivation $D : A \longrightarrow E$ is inner.

Proposition 4.1.16. *Let A be a completely contractive Banach algebra, and let E, F be operator A-bimodules. Then $E\widehat{\widehat{\otimes}}F$ becomes an operator A-bimodules through*

$$a \cdot (x \otimes y) = (a \cdot x) \otimes y \ \ and\ \ (x \otimes y) \cdot a = x \otimes (y \cdot a) \ \ (a \in A, x \in E, y \in F)$$

Definition 4.1.17. If A is a completely contractive Banach algebra, then the *diagonal operator* is a completely bounded A-bimodule homomorphism which is defined by

$$\Delta_A : A\widehat{\widehat{\otimes}}A \longrightarrow A, a \otimes b \mapsto ab$$

Definition 4.1.18. Let A be a completely contractive Banach algebra.

(a) An element $M \in (A\widehat{\widehat{\otimes}}A)^{**}$ is called a *virtual operator diagonal* for A if

$$a \cdot M = M \cdot a \text{ and } a \cdot \Delta_A^{**} M = a \ (a \in A)$$

(b) A bounded net $\{m_\alpha\}_{\alpha \in A} \subseteq A\widehat{\widehat{\otimes}}A$ is called an *approximate operator diagonal* for A if

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \text{ and } a\Delta_A m_\alpha \to 0 \ (a \in A)$$

Theorem 4.1.19. *An operator amenable completely contractive Banach algebra has a bounded approximate identity. For a completely contractive Banach algebra A, the followings are equivalent:*

(a) *A is operator amenable.*

(b) *There is a virtual operator diagonal for A.*

(c) *There is an approxiamte operator diagonal for A.*

Proof. Refer to [Rua1] [Theorem 2.4]. □

Definition 4.1.20. Let E be an operator space. A closed subspace F of E is said to be *completely weakly complemented* in E if there is a completely bounded projection from E^* onto $F^\perp$.

Theorem 4.1.21. *Let A be an operator amenable, completely contractive Banach algebra and let I be a closed ideal of A. Then the followings are equivalent:*

(a) *I is operator amenable.*

(b) *I has a bounded approximate identity.*

(c) *I is completely weakly complemented.*

Proof. Refer to [F-W] [Theorem 8.5] and [Wool] [Theorem 3]. □

4.2 Hopf-von Neumann algebras and their predual structures

Our sources of this section are [Rua2] and [Tak].

Definition 4.2.1. Given two von Neumann algebras $\mathfrak{M}$ and $\mathfrak{N}$ acting on Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ respectively, the *von Neumann algebra tensor product* $\mathfrak{M}\overline{\otimes}\mathfrak{N}$ is defined as the von Neumann algebraic acting on the Hilbert space tensor product $\mathcal{H} \otimes_2 \mathcal{K}$ generated by the algebra tensor product $\mathfrak{M} \otimes \mathfrak{N}$.

Theorem 4.2.2. *Let $\mathfrak{M}$, $\mathfrak{N}$ be von Neumann algebras with preduals $\mathfrak{M}_*$ and $\mathfrak{N}_*$, respectively. Then we have a canonical completely isometric isomorphism*

$$\mathfrak{M}_* \hat{\hat{\otimes}} \mathfrak{N}_* \cong (\mathfrak{M}\overline{\otimes}\mathfrak{N})_*$$

Proof. Refer to [Rua2] [Theorem 7.2.4]. □

Definition 4.2.3. A *Hopf-von Neumann algebra* is a pair $(\mathfrak{M}, \Gamma^*)$, where $\mathfrak{M}$ is a von Neumann algebra, and Γ^* is a *co-multiplication*, i.e. Γ^* is a unital, injective, normal, *-homomorphism from $\mathfrak{M}$ to $\mathfrak{M}\overline{\otimes}\mathfrak{M}$ which is *co-associative*:

i.e.

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "\Gamma^*"] \arrow[d, "\Gamma^*"] & \mathfrak{M}\overline{\otimes}\mathfrak{M} \arrow[d, "\Gamma^*\otimes \mathrm{id}_{\mathfrak{M}}"] \\
\mathfrak{M}\overline{\otimes}\mathfrak{M} \arrow[r, "\Gamma^*\otimes \mathrm{id}_{\mathfrak{M}}"] & \mathfrak{M}\overline{\otimes}\mathfrak{M}\overline{\otimes}\mathfrak{M}
\end{tikzcd}$$

Remarks.

(a) Let $\Gamma^* : \mathfrak{M} \longrightarrow \mathfrak{M}\overline{\otimes}\mathfrak{M}$ be a unital, injective, normal, *-homomorphism. Since Γ^* is w*-continuous, $\Gamma^* = (\Gamma)*$ for some $\Gamma : (\mathfrak{M}\overline{\otimes}\mathfrak{M})_* \longrightarrow (\mathfrak{M})_*$. Note that Γ^* is a *-homomorphism, Γ^* is completely contractive, so is $\Gamma : \mathfrak{M}_*\hat{\hat{\otimes}}\mathfrak{N}_* \cong (\mathfrak{M}\overline{\otimes}\mathfrak{M})_* \longrightarrow (\mathfrak{M})_*$. Thus the bilinear map $\Gamma : \mathfrak{M}_* \times \mathfrak{N}_* \longrightarrow (\mathfrak{M})_*$ is completely contractive. For any $f, g \in \mathfrak{M}_*$, define

$$f \circ_\Gamma g := \Gamma(f, g)$$

(b) $< f \otimes g \otimes h, (\Gamma^* \otimes \mathfrak{id}_{\mathfrak{M}})(\Gamma^*\phi) >$

$=< (\Gamma \otimes \mathfrak{id}_{\mathfrak{M}_*})(f \otimes g \otimes h), \Gamma^*\phi >$

$=< (f \circ_\Gamma g) \otimes h, \Gamma^*\phi >$

$=< \Gamma((f \circ_\Gamma g) \otimes h), \phi >$

$=< (f \circ_\Gamma g) \circ_\Gamma h, \phi >$

Similarly, $< f \otimes g \otimes h, (\mathfrak{id}_{\mathfrak{M}} \otimes \Gamma^*)(\Gamma^*\phi) > = < f \circ_\Gamma (g \circ_\Gamma h), \phi >$

Therefore, $\circ_\Gamma$ is associative if and only if Γ^* is co-associative. Thus, $(\mathfrak{M}_*, \circ_\Gamma)$ is a completely contractive Banach algebra provide that $(\mathfrak{M}, \Gamma^*)$ is a Hopf-von Neumann algebra.

4.3 Operator Cohomological properties of $A(G)$

Of course, someone may ask if there is any analogue for the Johnson's cohomology vanishing theorem for Fourier algebras. As proved in section 3.1, the amenability of the group does not imply the amenability of the Fourier algebra but only the existence of bounded approximate identity. In fact, $A(G)$ is amenable if and only if G has an abelian subgroup of finite index [F-R]. Therefore, the original definition of amenability is considered as "unsuitable" for the study of Fourier algebras. In [Rua1], Ruan, however, found a good notion of "amenability" for the Fourier algebra which is called "operator amenability". He proved that G is amenable if and only if $A(G)$ is operator amenable.

Let G be a locally compact group. From now on, use the following notations:

- Define $\Gamma^*_{VN(G)} : VN(G) \longrightarrow VN(G)\overline{\otimes}VN(G) \cong VN(G \times G)$,

$$T \mapsto W^*(T \otimes \mathfrak{id}_{L^2(G)})W$$

- The *fundamental operator* $W \in \mathfrak{L}(L^2(G \times G))$ is defined through

$$Wf(x,y) := \xi(x,xy) \quad (f \in L^2(G \times G), x, y \in G)$$

- Let the *right regular representation* of G $\rho_2 : G \longrightarrow \mathfrak{L}(L^2(G))$, $\rho_2(x)(\xi) := \xi * \delta_{x^{-1}}$ $(\xi \in L^2(G)))$ be defined via: $(\rho_2(x)(\xi))(y) := \Delta(x)^{\frac{1}{2}}\xi(yx)$ $(\xi \in L^2(G), x, y \in G)$

- For any unit vector $\xi \in L^2(G)$, define

$$m_\xi : G \times G \longrightarrow \mathbb{C}, (x,y) \mapsto < \lambda_2(x)\rho_2(y)\xi, \xi >$$

- Define $V \in \mathfrak{L}(L^2(G))$ through

$$(V\xi)(x) := \Delta(x)^{-\frac{1}{2}}\xi(x^{-1}) \quad (\xi \in L^2(G))$$

Proposition 4.3.1. *Let G be a locally compact group. Then*

(a) *W is a unitary and*

$$W^*f(x,y) = W^{-1}f(x,y) = f(x, x^{-1}y) \quad (x, y \in G)$$

(b) *For any unit vector $\xi \in L^2(G)$, $m_\xi \in B(G \times G)$*

(c) *V is self-adjoint and*

$$V^*\lambda_2(x)V = \rho_2(x) \quad (x \in G)$$

(d) $(V^* \otimes \mathfrak{id}_{L^2(G)})W^* = W(V^* \otimes \mathfrak{id}_{L^2(G)})$

(e) $(\rho_2(x) \otimes \mathfrak{id}_{L^2(G)})W = W(\rho_2(x) \otimes \mathfrak{id}_{L^2(G)})$ *for all $x \in G$*

(f) $\Gamma^*_{VN(G)}(\lambda_2(x)) = W^*(T \otimes \mathfrak{id}_{L^2(G)})W = \lambda_2(x) \otimes \lambda_2(x)$ *for all $x \in G$*

Proof. **(a)** :

$$\int_G(\int_G |W\xi(x,y)|^2 dx)dy$$

$$= \int_G(\int_G |\xi(x,xy)|^2 dx)dy = \int_G(\int_G |\xi(x,xy)|^2 dy)dx = \int_G(\int_G |\xi(x,y)|^2 dx)dy$$

implies that W is a unitary.

Therefore

$$W^*\xi(x,y) = W^{-1}\xi(x,y) = \xi(x,x^{-1}y)$$

(b) : Let $\pi_{\lambda_2,\rho_2} : G \times G \longrightarrow \mathfrak{L}(L^2(G)),\ (g,h) \mapsto \lambda_2(g)\rho_2(h)$. Then $\pi_{\lambda_2,\rho_2} \in \sum_{G\times G}$, and thus $m_\xi(g,h) =< \pi_{\lambda_2,\rho_2}(g,h)\xi, \xi >$ is an element in $B(G\times G)$ for all unit vector $\xi \in L^2(G)$.

(c): For any $\xi, \eta \in L^2(G)$,

$< \xi, V^*\eta >=< V\xi, \eta >$
$= \int_G \Delta(x)^{-\frac{1}{2}}\xi(x^{-1})\overline{\eta(x)}dm_G(x)$
$= \int_G \Delta(x)^{\frac{1}{2}}\xi(x)\overline{\eta(x^{-1})}dm_G(x^{-1})$
$= \int_G \Delta(x)^{\frac{1}{2}}\xi(x)\overline{\eta(x^{-1})}\Delta(x^{-1})dm_G(x)$
$= \int_G \xi(x)\overline{\eta(x^{-1})}\Delta(x)^{-\frac{1}{2}}dm_G(x)$

It follows that $V^*\eta(x) = V\eta(x^{-1})\Delta(x)^{-\frac{1}{2}} = \eta(x)$

For the second identity, let $\xi \in L^2(G), x, y \in G$. Then

$(V^*\lambda_2(x)V)(\xi(y))$
$= (V^*\lambda_2(x))(\Delta(y)^{-\frac{1}{2}}\eta(y^{-1}))$
$= V^*(\Delta(x^{-1}y)^{-\frac{1}{2}}\eta(y^{-1}x))$
$= \Delta(y)^{-\frac{1}{2}}\Delta(x^{-1}y^{-1})^{-\frac{1}{2}}\eta(yx))$
$= \Delta(x)^{\frac{1}{2}}\eta(yx)) = (\rho_2(x)\eta)(y)$

Therefore,, $V^*\lambda_2(x)V = \rho_2(x)$.

(d): For any $\xi \in L^2(G \times G)$, $x, y \in G$,

$(V^* \otimes \mathfrak{id}_{L^2(G)})W^*)(\xi(x,y))$
$= (V^* \otimes \mathfrak{id}_{L^2(G)})(\xi(x, x^{--1}y))$
$= \Delta(x)^{-\frac{1}{2}}\xi(x^{-1}, xy)$

$= W(\Delta(x)^{-\frac{1}{2}}\xi(x^{-1}, y))$

$= W(V^* \otimes \mathfrak{id}_{L^2(G)})(\xi(x, y))$

(e): Let $x \in G$, $f \in L^2(G \times G)$. Then

$W(\mathfrak{id}_{L^2(G)} \otimes \rho_2(x))f(y, z)$

$= W(\Delta(x)^{1/2} f(y, zx))$

$= \Delta(x)^{1/2} f(y, yzx)$

$= (\mathfrak{id}_{L^2(G)} \otimes \rho_2(x))f(y, yz)$

$= (\mathfrak{id}_{L^2(G)} \otimes \rho_2(x))Wf(y, z)$

(f): Let $x \in G$, $f \in L^2(G \times G)$. Then

$[\Gamma^*(\lambda_2(x))](f(y, z))$

$= [W^*(\lambda_2(x) \otimes \mathfrak{id}_{L^2(G)})]W(f(y, z))$

$= W^*(\lambda_2(x) \otimes \mathfrak{id}_{L^2(G)})(f(x, xz))$

$= W^*(f(x^{-1}y, (x^{-1}y)z))$

$= f(x^{-1}y, (x^{-1}y)(y^{-1}z))$

$= f(x^{-1}y, x^{-1}z)$

$= (\lambda_2(x) \otimes \lambda_2(x))(f(y, z))$

So, $\Gamma^*(\lambda_2(x)) = \lambda_2(x) \otimes \lambda_2(x)$ $(x \in G)$

□

Proposition 4.3.2. *With above notations, $(VN(G), \Gamma^*_{VN(G)})$ is a Hopf-von Neumann algebra and hence it induces a natural completely contractive Banach algebra structure on $A(G)$. Moreover, the multiplication $\circ_{\Gamma^*_{VN(G)}}$ is just the usual pointwise product of $A(G)$.*

Proof. We simply write $\Gamma^* := \Gamma^*_{VN(G)}$ for convenience.

Clearly, Γ^* is a unital, injective, normal, *-homomorphism.

Let $f_1, f_2 \in A(G), x \in G$. Then

$< f_1 \circ_{\Gamma^*} f_2, \lambda(x) >$

$=< f_1 \otimes f_2, \Gamma^*(\lambda(x)) >$

$=< f_1 \otimes f_2, \lambda_2(x) \otimes \lambda_2(x) >$

$=< f_1, \lambda_2(x) >< f_2, \lambda_2(x) >$

$= f_1(x^{-1}) f_2(x^{-1})$

$= f_1 \cdot f_2(x^{-1})$

$=< f_1 \cdot f_2, \lambda_2(x) >$

Since $\lambda_2(G)$ is dense in $VN(G)$, $\circ_{\Gamma^*}$ is just the pointwise product on $A(G)$. Therefore Γ^* is co-associative, and hence a comultiplication of $VN(G)$. □

Theorem 4.3.3. *Let G, H be two locally compact groups. Then*

$$A(G \times H) \cong_{c.i.} A(G)\hat{\hat{\otimes}}A(H)$$

Proof. Refer to [Rua2] [Section 16.2]. □

For complex functions u, v on G, write $u \odot v$ the function on $G \times G$ which is given by

$$(u \odot v)(x, y) := u(x)v(y) \quad (x, y \in G)$$

Proposition 4.3.4. *Let G be a locally compact group. Then we have:*

(a) *The $A(G)$-bimodule actions of $A(G \times G)$ induced by the identification $A(G)\hat{\hat{\otimes}}A(G) \cong A(G \times G)$ is given by*

$$(f \cdot g)(x, y) = f(x)g(x, y) \quad (f \in A(G),\, g \in A(G \times G),\, x, y \in G)$$

and

$$(g \cdot f)(x, y) = g(x, y)f(y) \quad (f \in A(G),\, g \in A(G \times G),\, x, y \in G)$$

Hence, these $A(G)$-bimodule actions of $A(G \times G)$ extend to $B(G \times G)$ in a canonical fashion.

(b) *Define $\tilde{\Delta} : B(G \times G) \longrightarrow B(G)$, $(\tilde{\Delta} f)(x) := f(x, x)$ $(x \in G)$. Then it extends the diagonal operator $\Delta : A(G \times G) \cong A(G)\hat{\hat{\otimes}}A(G) \longrightarrow A(G)$.*

Proof. Let $i : A(G)\widehat{\widehat{\otimes}}A(G) \longrightarrow A(G \times G)$, $i(f_1 \otimes f_2) = f_1 \odot f_2$ be the canonical complete isometric isomorphism. Clearly, $A(G) \odot A(G) = i(A(G) \otimes A(G))$ is dense in $A(G \times G)$.
(a): Note that the natural $A(G)$-bimodule actions on $A(G) \otimes A(G)$ is given by

$$(f_1 \otimes f_2) \cdot g = f_1 \otimes f_2 g \quad \textit{and} \quad g \cdot (f_1 \otimes f_2) = g f_1 \otimes f_2 \quad (f_1, f_2, g \in A(G))$$

Thus, the $A(G)$-bimodule actions induced on $A(G) \odot A(G)$ are given by

$$(f_1 \odot f_2) \cdot g = f_1 \odot f_2 g \quad (f_1, f_2, g \in A(G))$$

and

$$g \cdot (f_1 \odot f_2) = g f_1 \odot f_2 \quad (f_1, f_2, g \in A(G))$$

Clearly, these actions are continuous and extends to the actions on $A(G \times G)$ stated in the theorem. Therefore, it extends to the actions on $B(G \times G)$ via the same formulae. In addition, for any $f \in A(G), g \in B(G \times G)$,

$$\|f \cdot g\|_{B(G\times G)} \leq \|f \odot 1\|_{B(G\times G)} \|g\|_{B(G\times G)} \leq \|f\|_{A(G)} \|g\|_{B(G\times G)}$$

and

$$\|g \cdot f\|_{B(G\times G)} \leq \|g\|_{B(G\times G)} \|f \odot 1\|_{B(G\times G)} \leq \|g\|_{B(G\times G)} \|f\|_{A(G)}$$

The results follows.
(b): Note that the corresponding mapping induced by the isometric isomorphism i : $A(G)\widehat{\widehat{\otimes}}A(G) \longrightarrow A(G \times G)$ is given by

$$\bar{\Delta} : A(G) \odot A(G) \longrightarrow A(G) \ , \ \bar{\Delta}(f \odot g)(x) = (f \odot g)(x, x) = f(x) g(x)$$

Define $\Delta : A(G \times G) \longrightarrow A(G)$,$\Delta(h)(x) = h(x, x)$. then

$$\Delta(f_1 \odot f_2))(x) = f_1(x) f_2(x) = \bar{\Delta}(f_1 \odot f_2)(x) \quad (f_1, f_2 \in A(G), x \in G)$$

Since Δ and $\bar{\Delta}$ agree on the dense subset $A(G) \odot A(G)$, they are equal on $A(G \times G)$. $\square$

Lemma 4.3.5. *Let G be an amenable locally compact group, and suppose that there is a bounded net $(m_\alpha)_{\alpha \in A}$ in $B(G \times G)$ such that*

$$\|f \cdot m_\alpha - m_\alpha \cdot f\|_{B(G\times G)} \to 0 \ \ (f \in A(G))$$

and

$$\|f\Delta m_\alpha - f\|_{B(G)} \to 0 \ \ (f \in A(G))$$

Then $A(G)$ is operator amenable.

Proof. Let $(e_\beta)_{\beta \in B}$ be a bounded approximate identity of $A(G)$ such that $sup_{\beta\in B}\|e_\beta\| = 1$. Since $A(G\times G)$ is a closed ideal of $B(G\times G)$, the net $(e_\beta \cdot m_\alpha \cdot e_\beta)_{(\alpha,\beta)\in A\times B}$ lies in $A(G\times G)$. For any $f \in A(G)$,

$$[f \cdot (e_\beta \cdot m_\alpha \cdot e_\beta) - (e_\beta \cdot m_\alpha \cdot e_\beta) \cdot f](x, y)$$

$$= f(x)e_\beta(x)m_\alpha(x, y)e_\beta(y) - e_\beta(x)m_\alpha(x, y)e_\beta(y)f(y)$$

$$= e_\beta(x)e_\beta(y)[f(x)m_\alpha(x, y) - m_\alpha(x, y)f(y)]$$

So,

$$\|f \cdot (e_\beta \cdot m_\alpha \cdot e_\beta) - (e_\beta \cdot m_\alpha \cdot e_\beta) \cdot f\|_{A(G\times G)} \le \|e_\beta \odot e_\beta\|_{A(G\times G)}\|f \cdot m_\alpha - m_\alpha \cdot f\|_{B(G\times G)} \to 0$$

and

$$f(x)\Delta(e_\beta \cdot m_\alpha \cdot e_\beta)(x) - f(x) = e_\beta(x)^2 f(x)m_\alpha(x) - f(x)$$

implies that

$$\|f\Delta(e_\beta \cdot m_\alpha \cdot e_\beta) - f\|_{A(G)}$$

$$= \|e_\beta^2 f\Delta m_\alpha - f\|_{A(G)} \le \|e_\beta\|^2\|f\|\|\Delta m_\alpha - f\|_{A(G)} + \|e_\beta^2 f - f\|_{A(G)} \to 0$$

□

Lemma 4.3.6. *Let G be a locally compact group, and suppose that there is a net $(\xi_\alpha)_{\alpha\in A}$ of unit vectors in $L^2(G)$ satisfying*

$$\|W(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta\| \to 0 \ \ (\eta \in L^2(G))$$

and

$$\|\lambda_2(x)\rho_2(x)\xi_\alpha - \xi_\alpha\| \to 0$$

uniformly on all compact subsets of G. Then the net $(m_{\xi_\alpha})_\alpha$ in $B(G \times G)$ satisfies the hypotheses of lemma 4.3.5.

Proof. Let $f \in A(G)$. By the polarization identity, we may suppose that

$$f(x) = < \lambda_2(x)\eta, \eta > \quad (x \in G)$$

for some $\eta \in L^2(G)$. Note that for any $x, y \in G$,

$$\begin{aligned}
&(f \cdot m_{\xi_\alpha})(x,y) \\
&= < \lambda_2(x)\eta, \eta >< \lambda_2(x)\rho(y)\xi_\alpha, \xi_\alpha > \\
&= < \lambda_2(x)\eta, \eta >< \lambda_2(x)V^*\lambda_2(y)V\xi_\alpha, \xi_\alpha > \text{ by lemma 4.3.1(c)} \\
&= < (\lambda_2(x)V^*\lambda_2(y)V \otimes \lambda_2(x))(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta >
\end{aligned}$$

and

$$(m_{\xi_\alpha} \cdot f)(x,y)$$

$$= < \lambda_2(x)\rho(y)\xi_\alpha, \xi_\alpha >< \lambda_2(y)\eta, \eta >$$

$$= < \lambda_2(x)V^*\lambda_2(y)V\xi_\alpha, \xi_\alpha >< \lambda_2(y)\eta, \eta > \text{ by lemma 4.3.1(c)}$$

$$= < (\lambda_2(x) \otimes id_{L^2(G)})(V^*\lambda_2(y)V \otimes \lambda_2(y))(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta >$$

$$= < (\lambda_2(x) \otimes id_{L^2(G)})(V^* \otimes id_{L^2(G)})(\lambda_2(y) \otimes \lambda_2(y))(V \otimes id_{L^2(G)})(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta >$$

$$= < (\lambda_2(x) \otimes id_{L^2(G)})(V^* \otimes id_{L^2(G)})W^*(\lambda_2(y) \otimes id_{L^2(G)})W(V \otimes id_{L^2(G)})(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta >$$

by lemma 4.3.1(f)

$$= < (\lambda_2(x) \otimes id_{L^2(G)})W(V^* \otimes id_{L^2(G)})(\lambda_2(y) \otimes id_{L^2(G)})(V \otimes id_{L^2(G)})W^*(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta >$$

by lemma 4.3.1(c), (d)

$$= < W^*(\lambda_2(x) \otimes id_{L^2(G)})W(V^* \otimes id_{L^2(G)})(\lambda_2(y) \otimes id_{L^2(G)})(V \otimes id_{L^2(G)})W^*(\xi_\alpha \otimes \eta), W^*(\xi_\alpha \otimes \eta) >$$

$$= < (\lambda_2(x) \otimes \lambda_2(x))(V^*\lambda_2(y)V \otimes id_{L^2(G)})W^*(\xi_\alpha \otimes \eta), W^*(\xi_\alpha \otimes \eta) >$$

$$= < (\lambda_2(x)V^*\lambda_2(y)V \otimes \lambda_2(x))W^*(\xi_\alpha \otimes \eta), W^*(\xi_\alpha \otimes \eta) >$$

Then we obtain

$$(f \cdot m_{\xi_\alpha} - m_{\xi_\alpha} \cdot f)(x,y) =< (\lambda_2(x)V^*\lambda_2(y)V \otimes \lambda_2(x))(\xi_\alpha \otimes \eta - W^*(\xi_\alpha \otimes \eta)), \xi_\alpha \otimes \eta >$$

$$+ < (\lambda_2(x)V^*\lambda_2(y)V \otimes \lambda_2(x))(W^*(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta), W^*(\xi_\alpha \otimes \eta) >$$

Therefore,

$$\|f \cdot m_{\xi_\alpha} - m_{\xi_\alpha} \cdot f\|_{B(G\times G)} \leq 2\|\eta\|\|\xi_\alpha \otimes \eta - W^*(\xi_\alpha \otimes \eta)\| = 2\|\eta\|\|W(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta\| \to 0$$

Since $\{x \mapsto < \lambda_2(x)\eta, \eta >: \eta \in C_c(G)\} = C_C(G) * C_c(G)^\sim$ is dense in $A(G)$, we may suppose that

$$f(x) =< \lambda_2(x)\eta, \eta > \quad (x \in G)$$

for some $\eta \in C_c(G)$.

Note that for any $x \in G$, we have

$$\begin{aligned}
&(f\Delta m_{\xi_\alpha})(x) \\
&= f(x)m_{\xi_\alpha}(x,x) \\
&=< \lambda_2(x)\rho(x)\xi_\alpha, \xi_\alpha >< \lambda_2(x)\eta, \eta > \\
&=< \lambda_2(x)V^*\lambda_2(x)V\xi_\alpha, \xi_\alpha >< \lambda_2(x)\eta, \eta > \text{ by lemma 4.3.1(c)} \\
&=< (\lambda_2(x) \otimes \lambda_2(x))(id_{L^2(G)} \otimes V^*\lambda_2(x)V)(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta > \\
&=< W^*(\lambda_2(x) \otimes id_{L^2(G)})W(id_{L^2(G)} \otimes V^*\lambda_2(x)V)(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta > \text{ by lemma 4.3.1(f)} \\
&=< W^*(\lambda_2(x) \otimes id_{L^2(G)})(id_{L^2(G)} \otimes V^*\lambda_2(x)V)W(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta > \text{ by lemma 4.3.1(e)} \\
&=< W^*(id_{L^2(G)} \otimes V^*)(\lambda_2(x) \otimes \lambda_2(x))(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta > \\
&=< W^*(id_{L^2(G)} \otimes V^*)W^*(\lambda_2(x) \otimes id_{L^2(G)})W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta), \xi_\alpha \otimes \eta > \text{ by lemma 4.3.1(f)} \\
&=< (\lambda_2(x) \otimes id_{L^2(G)})W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta), W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta) >
\end{aligned}$$

It follows that

$$(f\Delta m_{\xi_\alpha} - f)(x) =< (\lambda_2(x) \otimes id_{L^2(G)})(W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta), W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta) >$$

$$+ < (\lambda_2(x) \otimes id_{L^2(G)})(\xi_\alpha \otimes \eta), W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta >$$

Therefore, we have

$$\|f\Delta(m_{\xi_\alpha}) - f\| \leq 2\|\eta\|\|W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta\|$$

Let $K := \text{supp}\eta$. Since $\|\lambda_2(x)\rho_2(x)\xi_\alpha - \xi_\alpha\|_2 \to 0$ uniformly on K, we obtain:

$$\begin{aligned}
&\|W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta\| \\
&= \int_G \int_G |W(id_{L^2(G)} \otimes V)W(\xi_\alpha \otimes \eta)(x,y) - (\xi_\alpha \otimes \eta)(x,y)|dydx \\
&= \int_G \int_G |\Delta(x)^{-1/2}\eta(x)\xi_\alpha(xyx^{-1}) - \eta(x)\xi_\alpha(y)|^2 dydx \\
&= \int_K |\eta(x)|^2 \int_G |\Delta(x)^{-1/2}\xi_\alpha(xyx^{-1}) - \xi_\alpha(y)|^2 dydx \\
&= \int_K |\eta(x)|^2 \|\lambda_2(x)\rho_2(x)\xi_\alpha - \xi_\alpha\|_2^2 dx \to 0
\end{aligned}$$

So, $\|f\Delta(m_{\xi_\alpha}) - f\| \to 0$. □

Theorem 4.3.7. *Let G be a locally compact group, and let $A(G)$ be its Fourier algebra. Then the following statements are equivalent:*

(a) *G is amenable.*

(b) *$A(G)$ is operator amenable.*

Proof. "(a) ⇒ (b)" See [Rua1] or [Run] [Chapter 7].

"(b) ⇒ (a)" By theorem 4.1.19, $A(G)$ has a bounded approximate identity. Therefore, G is amenable by theorem 3.2.2. □

4.4 Ideals in $A(G)$ with bounded approximate identities

We shall see in this section that the amenability of G can be characterized by the existence of bounded approximate identity in certain ideals of $A(G)$: G is amenable if and only if $I(H)$(defined below)$\trianglelefteq A(G)$ has a bounded approximate identity for any subgroup H in G.

Definition 4.4.1. Let G be a locally compact group.

(a) For any $E \subseteq G$, a closed ideal $I(E)$ of $A(G)$ is defined by

$$I(E) := \{u \in A(G) : u(x) = 0 \ for \ all \ x \in E\}$$

(b) For any closed ideal $I \trianglelefteq A(G)$, define

$$I^{\perp} := \{T \in VN(G) :< T, u >= 0 \ for \ each \ u \in I\}$$

(c) For any subgroup H of G, define $VN_H(G)$ to be the von Neumann algebra of $VN(G)$ generated by $\{\lambda_2(x) : x \in H\}$.

Lemma 4.4.2. *Let G be a locally compact group and H a closed subgroup of G. For any $v \in A(H)$, there exists $u \in A(G)$ such that $u|_H = v$ and $\|v\|_{A(H)} = inf\{\|u\|_{A(G)} : u|_H = v\}$*

Proof. Refer to [Her] [Theorem 1]. □

Theorem 4.4.3. *Let G be a locally compact group. Then we have:*

(a) *For any closed ideal $I \trianglelefteq A(G)$, $I^{\perp}$ is an $A(G)$-submodule of $VN(G)$.*

(b) *For each closed subgroup H of G, $A(G)/I(H)$ is isometrically isomorphic to $A(H)$. Hence, $I(H)^{\perp}$ is isometrically isomorphic to $VN_H(G)$.*

(c) *$VN_H(G)$ is *-isomorphic to $VN(H)$.*

Proof. Refer to [For2] [Lemma 3.8] and [F-W] [Proposition 4.2, 4.3]. □

Lemma 4.4.4. *Let H be an amenable locally compact group. Then there exists a completely contractive projection $P : \mathfrak{L}(L^2(H)) \longrightarrow VN(H)$.*

Proof. Note that $VN(H)$ is the commutant of $\{\rho_2(x) : x \in H\}$. Let m be a left invariant mean on $L^{\infty}(H)$. For convenience, we regard m as a finitely additive measure on H. Now define $P : \mathfrak{L}(L^2(H)) \longrightarrow \mathfrak{L}(L^2(H))$ to be the weak operator converging integral

$$P(T) = \int_H \rho_2(x) T \rho_2(x)^* dm_H(x)$$

for each $T \in \mathfrak{L}(L^2(H))$. That is,

$$< P(T)f, g >= \int_H < \rho_2(x)T\rho_2(x)^* f, g > dm_H(x)$$

for each $f, g \in L^2(H)$. From the invariance of m, it is easy to see that for each $T \in \mathfrak{L}(L^2(H))$,

$$\rho_2(x)P(T) = P(T)\rho_2(x)$$

for all $x \in H$. It follows that $P(T) \in VN(H)$. It is also clear that if $T \in VN(H)$, then $P(T) \in VN(H)$.

We have that P is completely positive, since each amplification $P_n : \mathbb{M}_n(\mathfrak{L}(L^2(H))) \longrightarrow \mathbb{M}_n(VN(H))$ is given by the weak operator converging integral

$$P_n([T_{ij}]) = \int_H diag(\rho_2(x))\,[T_{ij}]\,diag(\rho_2(x))^* dm_H(x)$$

for each $[T_{ij}] \in \mathbb{M}_n(\mathfrak{L}(L^2(H)))$, where $diag(\rho_2(x))$ denotes the $n \times n$ diagonal matrix with all diagonal entries equal to $\rho_2(x)$. Finally, since $P(I) = I$, P is completely contractive.

□

Proposition 4.4.5. *Let G be a locally compact group and let H be an amenable closed subgroup of G. Then there exists a completely contractive projection from $VN(G)$ onto $I(H)^\perp$.*

Proof. By lemma 4.4.4, there exists a completely contractive projection $P : \mathfrak{L}(L^2(H)) \longrightarrow VN(H)$. let $\Phi : VN(H) \longrightarrow VN_H(G) \subseteq \mathfrak{L}(L^2(G))$ be a *-isomorphism. By the Arveson-Wittstock extension theorem,

$$\Phi^{-1} : VN_H(G) \longrightarrow VN(H)$$

has a completely contractive extension

$$\Psi : \mathfrak{L}(L^2(G)) \longrightarrow \mathfrak{L}(L^2(H))$$

Since $\Psi(I) = I$, Ψ is completely positive by lemma 4.1.7.

Let

$$P := \Phi \circ P \circ \Psi : \mathfrak{L}(L^2(G)) \longrightarrow VN_H(G) = I(H)^\perp$$

Then $Q := P|_{VN(G)}$ is the required projection. □

Theorem 4.4.6. *Let G be a locally compact group. Then the following statements are equivalent:*

(a) *G is amenable.*

(b) *$I(H)$ has a bounded approximate identity for any closed proper subgroup H of G.*

(c) *$I(H)$ has a bounded approximate identity for some closed amenable subgroup H of G.*

(d) *$I(H)$ has a bounded approximate identity for some closed proper subgroup H of G.*

Proof.
"(a) $\Rightarrow$ (b)" Suppose that G is amenable, then $A(G)$ is operator amenable. For any closed subgroup H, H is amenable by theorem 2.1.6. By proposition 4.4.5, $I(H)$ is completely weakly complemented in the operator amenable completely contractive Banach algebra $A(G)$. Therefore, $I(H)$ has a bounded approximate identity by theorem 4.1.21.

"(b) $\Rightarrow$ (c)", "(b) $\Rightarrow$ (d)" Trivial.

"(c) $\Rightarrow$ (a)" Assume that H is an amenable closed subgroup of G such that $I(H)$ has a bounded approximate identity. Theorem 3.2.2 and theorem 4.4.3 imply that $A(G)/I(H) \cong A(H)$ has a bounded approximate identity. If $A(G)/I(H)$ and $I(H)$ both have bounded approximate identity, then it is easy to construct a bounded approximate identity for $A(G)$.

"(d) $\Rightarrow$ (a)" Since H is proper, there exists $x \in G \setminus H$. Furthermore $I(xH)$ has a bounded approximate identity $\{u_\alpha\}_{\alpha \in A}$. Assume that $v \in C_c(H) \cap A(H)$ with supp$v = K \subseteq H$. We can find a neighborhood V of K in G and a $u \in B(G)$ such that $u(K) = 1$, supp$u \subset V$ and $V \cap xH = \phi$. By lemma 4.4.2, there exists $u_1 \in A(G)$ such that $u_1|_H = v$.

Let $w = u_1 u$. Then $w \in I(xH)$ and $w|_H = v$. let $v_\alpha = u_\alpha|_H$. Then $v_\alpha \in A(H)$ and $\|v_\alpha\|_{A(H)} \leq \|u_\alpha\|_{A(G)}$ by lemma 4.4.2. Therefore, for any $v \in C_c(H) \cap A(H)$ and $w \in I(xH)$ with $w|_H = v$,

$$0 \leq \lim_\alpha \|v_\alpha v - v\|_{A(H)} \leq \lim_\alpha \|u_\alpha w - w\|_{A(G)} = 0$$

Since $A(H) \cap C_c(H)$ is dense in $A(H)$, $\{v_\alpha\}_{a \in A}$ is a bounded approximate identity in $A(H)$. It follows that H is amenable. Therefore, $I(H)$ has a bounded approximate identity for some closed amenable subgroup H of G. By "(c) $\Rightarrow$ (a)", G is amenable. $\square$

Remark. Lemma 4.4.4 and lemma 4.4.5 are [F-K-L-S] [lemma 1.1] and [Proposition 1.2] respectively. Theorem 4.4.6 is a mixture of [F-K-L-S] [Corollary 1.6], [For2] [Theorem 3.9] and [For3] [Theorem 3.9].

4.5 Other Cohomological properties of $A(G)$

As we shall see in this section, other evidences for the suitability of the theory of operator spaces for the study of Fourier algebras are: $A(G)$ is always operator weakly amenable (compare with theorem 2.1.9) and $A(G)$ is operator biprojective if and only if G is discrete (compare with theorem 2.1.12).

For a completely contractive Banach algebra A, let $M_A : A^\sharp \hat{\hat{\otimes}} A \longrightarrow A, a \otimes b \mapsto a \cdot b$ be the left module multiplication map of the $A^\sharp$-bimodule A, and let $m_A : A^\sharp \hat{\hat{\otimes}} A^\sharp \longrightarrow A^\sharp$ be the multiplication map. Denote:

(a) $K^A := Ker(M_A)$

(b) $[K; A] := \overline{span\{a \cdot u - u \cdot a : u \in K, a \in A\}}$

(c) $K_1^A := Ker(m_A)$

(d) $K_0^A := K_1 \cap (A \hat{\hat{\otimes}} A)$

Remark. It is not hard to see that $K^A := K_1^A \cap (A^\sharp \hat{\hat{\otimes}} A)$.

Definition 4.5.1. A completely contractive Banach algebra A is said to be *operator weakly amenable* if every completely bounded derivation $D : A \longrightarrow A^*$ is inner.

Theorem 4.5.2. *Let A be a commutative completely contractive Banach algebra. Then A is operator weakly amenable if and only if $\overline{A^2} = A$ and $\overline{(K_0^A)^2} = \overline{(A\widehat{\widehat{\otimes}}A) \cdot K_1^A}$.*

Proof. Refer to [Spr] [Theorem 2.2]. □

Let G be a locally compact group and let $\pi \in \sum_G$. Denote by $1 : G \longrightarrow \mathbb{C}$ the constant function or the trivial representation of G, put $A_\pi := \overline{span\{x \mapsto < \pi(x), \xi, \eta >: \xi, \eta \in H_\pi\}}^{\|\cdot\|_{B(G)}}$ and write VN_π the von Neumann algebra generated by $\pi(G)$ in $\mathfrak{L}(\mathcal{H}\pi)$.

Definition 4.5.3. Let G be a locally compact group and let π, $\sigma \in \sum_G$.

(a) Let $\pi \times \sigma : G \times G \longrightarrow \mathcal{U}(\mathcal{H}_\pi \times H_\rho)$ be the *Kronecker product* of π and σ, given by

$$(\pi \times \sigma)(s,t) = \pi(s) \otimes \sigma(t) \quad (s,t \in G)$$

(b) π and σ are said to be *disjoint* if they do not have subrepresentations which are unitary equivalent.

Theorem 4.5.4. *Let G be a locally compact group and let π, $\sigma \in \sum_G$. Then we have:*

(a) $(A_\pi, \|\cdot\|_{B(G)})^* \cong (VN_\pi, \|\cdot\|_{\mathfrak{L}(\mathcal{H}_\pi)})$.

(b) $VN_\pi \overline{\otimes} VN_\sigma \cong VN_{\pi\times\sigma}$

(c) *If π and σ are disjoint, then $VN_{\pi\oplus\sigma} \cong VN_\pi \oplus_\infty VN_\sigma$ and $A_{\pi\oplus\sigma} \cong A_\pi \oplus_1 A_\sigma$.*

Proof. Refer to [Spr] [Introduction of Section 2]. □

Theorem 4.5.5. *Let G be a locally compact group. Then the representations 1×1, $\lambda_2 \times 1$, $1 \times \lambda_2$, $\lambda_2 \times \lambda_2$ are all disjoint and we have the following complete isometric isomorphism:*

(a) $A(G)^\sharp \cong A_{\lambda_2 \oplus 1}$

(b) $A(G)^\sharp \widehat{\widehat{\otimes}} A(G) \cong A_{(\lambda_2\times\lambda_2)\oplus(1\times\lambda_2)} = span\{u, 1 \times v : u \in A(G \times G) \text{ and } v \in A(G)\}$

(c) $A(G)^\sharp \widehat{\widehat{\otimes}} A(G)^\sharp \cong A_{(\lambda_2\times\lambda_2)\oplus(1\times\lambda_2)\oplus(\lambda_2\times 1)\oplus(1\times 1)} = span\{u, 1\times v, v\times 1, 1\times 1 : u \in A(G\times G)$ *and* $v \in A(G)\}$

Proof. Refer to [Spr] [Prop 3.1, 3.2]. □

Theorem 4.5.6. *Let G be a locally compact group and let $A(G)$ be its Fourier algebra. Then $A(G)$ is always operator weakly amenable.*

Proof. If G is compact, then $A(G)$ is operator amenable and hence operator weakly amenable.

If G is non-compact, let $\pi := (\lambda_2 \times \lambda_2) \oplus (1 \times \lambda_2) \oplus (\lambda_2 \times 1) \oplus (1 \times 1)$ such that A_π is a subalgebra of $B(G \times G)$. Write $K_1 = K_1^{A(G)}$ and $K_0 = K_0^{A(G)}$.

In the identification $A(G)^\sharp \widehat{\widehat{\otimes}} A(G)^\sharp \cong A_\pi$, the multiplication map $m_{A(G)}$ corresponds to the map

$$R : A_\pi \longrightarrow A_{\lambda_2\oplus 1}, Ru(s) = u(s,s) \quad (s \in G)$$

Let $G_D := \{(s,s) : s \in G\}$ be the diagonal subgroup of $G \times G$. It is easily seen that $R(u) = u|_{G_D}$. Then

$$K_1 \cong Ker(R) = span\{u - 1 \times R(u), u - R(u) \times 1 : u \in A_\pi\}$$

and

$$K_0 \cong Ker(R) \cap A(G \times G) = I(G_D)$$

Note that $I(G_D)$ is a set of spectral synthesis.(See [Her]) We thus obtain that

$$\overline{K_0^2} \cong \overline{I(G_D)} = \overline{I(G_D)} = K_0 \quad (*)$$

Since $A(G \times G)$ is a closed ideal of A_π, we get

$$(A(G)\widehat{\widehat{\otimes}}A(G)) \cdot K_1 \cong A(G \times G) \cdot Ker(R) \subseteq A(G \times G) \cap Ker(R) = I(G_D) \cong K_0$$

On the other hand, by (*),

$$K_0 \cong I(G_D) = \overline{A(G \times G) \cdot I(G_D)} \subseteq \overline{A(G \times G) \cdot Ker(R)} \cong \overline{(A(G)\widehat{\widehat{\otimes}}A(G)) \cdot K_1}$$

We thus have

$$\overline{(A(G)\widehat{\widehat{\otimes}}A(G)) \cdot K_1} = K_0 = \overline{K_0^2}$$

By the Tauberian theorem for $A(G)$, $\overline{A(G)^2} = A(G)$. Hence, $A(G)$ is operator weakly amenable by theorem 4.5.2. □

Remark. Above lemmas and theorems can be founded in [Spr].

The following theorem can be considered as a "dual" result of theorem 2.1.12 and its proof can be founded in [Ari].

Lemma 4.5.7. *Let A be a commutative, operator biprojective, completely contractive Banach algebra. Then $\sigma(A)$, the spectrum of A, is discrete.*

Proof. Refer to [Ari] [Theorem 7.29, 7.30]. □

Theorem 4.5.8. *Let G be a locally compact group. Then G is discrete if and only if $A(G)$ is operator biprojective.*

Proof. Assume that G is discrete. Recall that the diagonal operator induced on $A(G \times G)$ is given by $\Delta : A(G \times G) \longrightarrow A(G)$, $\Delta(f(x)) := f(x, x)$ for any $x \in G$. For $f \in A(G)$, denote by $\rho(f)$ the function on $G \times G$ such that

$$\rho(f)(x, x) = f(x, x) \quad (x \in G)$$

and

$$\rho(f)(x, y) = 0 \quad (x, y \in G,\ x \neq y)$$

Let $f(x) =< \lambda_2(x)\xi, \eta >$ for some $\xi, \eta \in l^2(G)$ and let $\xi = \sum_{x \in G} \xi_x \chi_x$ and $\eta = \sum_{h \in G} \eta_y \chi_y$ where $(\xi_x)_{x \in G}$, $(\eta_y)_{y \in G} \in l^2$. Then

$$\rho(f)(x, y) =< \lambda_2(x) \otimes \lambda_2(x)\xi', \eta' >$$

where $\xi' = \sum_{x \in G} \xi_x \chi_{x,x}$ and $\eta' = \sum_{y \in G} \eta_y \chi_{y,y}$. Moreover, $\|\xi'\|_{l^2(G \times G)} = \|\xi\|_{l^2(G)}$ and $\|\eta'\|_{l^2(G \times G)} = \|\eta\|_{l^2(G)}$ whence $\|\rho(f)\|_{A(G \times G)} \leq \|f\|_{A(G)}$. Consequently, $\rho(f) \in A(G \times G)$.

It is easily seen from the definition that $\rho : A(G) \longrightarrow A(G \times G)$ is a homomorphism which is a right inverse of Δ. Conversely, since $\sigma(A(G)) = G$, the result follows by lemma 4.5.7.

$\square$

An alternative proof of this theorem can be found in [Woo2].

Appendix A

Objects or notions and their "dual" versions

OBJECTS AND NOTIONS	"DUAL" OBJECTS OR NOTIONS
G	$\widehat{G}$
$L^1(G)$	$A(G)$
$M(G)$	$B(G)$
$C_0(G)$	$C^*(G)$
$L^\infty(G)$	$VN(G)$
$UCB(G)$	$UCB(\widehat{G})$
$\|\cdot\|_\infty$	$\|\cdot\|_{C^*}$
$\|\cdot\|_{C^*}$	$\|\cdot\|_\infty$
Discrete group	Compact group
Compact group	Discrete group

Appendix B

Results and "dual results" for general locally compact groups

RESULTS	"DUAL" RESULTS
$L^1(G) \trianglelefteq M(G)$	$A(G) \trianglelefteq B(G)$
$L^1(G_1) \cong L^1(G_2)$ if and only if $G_1 \cong G_2$	$A(G_1) \cong A(G_2)$ if and only if $G_1 \cong G_2$
$M(G_1) \cong M(G_2)$ if and only if $G_1 \cong G_2$	$B(G_1) \cong B(G_2)$ if and only if $G_1 \cong G_2$
$L^1(G)^* \cong L^\infty(G)$	$A(G)^* \cong VN(G)$
$C_0(G)^* \cong M(G)$	$C^*(G)^* \cong B(G)$
$L^1(G)$ is $\|\cdot\|_{C^*(G)}$-dense in $C^*(G)$	$A(G)$ is $\|\cdot\|_\infty$-dense in $C_0(G)$
$L^1(G)$ is amenable if and only if G is amenable	$A(G)$ is operator-amenable if and only if G is amenable
$L^1(G)$ is weakly amenable	$A(G)$ is operator weakly amenable
$L^1(G)$ is biprojective if and only if G is compact	$A(G)$ is operator biprojective if and only if G is discrete
$L^1(G)$ is unital $\iff$ G is discrete $\iff$ $L^1(G) \cong M_a(G) = M(G)$	$A(G)$ is unital $\iff$ G is compact $\iff$ $A(G) = B(G)$
$L^1(G_1) \widehat{\otimes} L^1(G_2) \cong L^1(G_1 \times G_2)$	$A(G_1) \widehat{\widehat{\otimes}} A(G_2) \cong_{c.i.} A(G_1 \times G_2)$

Appendix C

Results and "dual results" which are equivalent to the amenability of the group

RESULTS	"DUAL RESULTS" WHICH ARE EQUIVALENT TO THE AMENABILITY OF THE GROUP
$L^1(G)$ has a bounded approximate identity	$A(G)$ has a bounded approximate identity
$C^*(G) = L^1(G) * C^*(G)$	$C_0(G) = A(G) \cdot C_0(G)$
$L^1(G)$ is closed in $\mathfrak{M}(L^1(G), L^1(G))$	$A(G)$ is closed in $\mathfrak{M}(A(G), A(G))$
$L^1(G)$ factors(weakly)	$A(G)$ factors(weakly)
$L^1(G)$ is $\sigma(M(G), C_0(G))$-dense in $M(G)$	$A(G)$ is $\sigma(B(G), C^*(G))$-dense in $B(G)$
$\mathfrak{M}(L^1(G), L^1(G)) \cong M(G)$	$\mathfrak{M}(A(G), A(G)) \cong B(G)$
$UCB(G) = L^1(G) * L^\infty(G) * L^1(G)$ is a closed subalgebra of $L^\infty(G)$	$A(G) \cdot VN(G)$ is a closed subalgebra of $VN(G)$

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